Stochastic Simulation and Monte Carlo Methods: Exercise Class 0.

Olivier Wintenberger: olivier.wintenberger@upmc.fr

Exercise 1. We recall that $X \sim \mathcal{G}eo(p)$ iff $X \in \mathbb{N}$ and $F_X(k) = 1 - (1-p)^k$, $k \in \mathbb{N}$.

1. Compute the density wrt the counting measure, the expectation and the variance of X,
2. Show that X is memoryless in the sense that
$$\mathbb{P}(X > k + n \mid X > n) = \mathbb{P}(X > k)\,, \qquad n, k \geq 0.$$
3. Prove the reciprocal: any $X \in \mathbb{N}$ memoryless is geometrically distributed.

Exercise 2. We recall that $X \sim \mathcal{E}xp(\lambda)$ iff $F_X(x) = 1 - \exp(-\lambda x)$, $x \geq 0$.

1. Compute the density wrt the Lebesgue measure, the expectation and the variance of X,
2. Show that X is memoryless in the sense that
$$\mathbb{P}(X > x + y \mid X > x) = \mathbb{P}(X > y)\,, \qquad x, y \geq 0.$$
3. Prove the reciprocal: any $X > 0$ memoryless is exponentially distributed.

Exercise 3. Let X, Y be two independent rv distributed as $\mathcal{P}oisson(\lambda_1)$ and $\mathcal{P}oisson(\lambda_2)$, respectively.

1. What is the distribution of X_1 given $X_1 + X_2$?
2. Deduce $\mathbb{E}[X_1 \mid X_1 + X_2]$.

Exercise 4. Let X, Y and Z be iid $\mathcal{E}xp(\lambda)$ rv. Determine the distribution of $(Y - X, Z - X)$ given X.

Exercise 5. We consider the linear model approximating Y with $aX + b$ where a is the slope and b is the intercept of the regression line. Let (X, Y) be continuous and admits a density $f_{X,Y}(x, y) = x^{-1} 1\!\!1_{0<y<x<1}$ wrt Lebesgue.

1. Compute the marginal densities f_X and f_Y.
2. Compute the distribution of Y given X and $\mathbb{E}[X \mid Y]$.
3. Determine the coefficients (a, b) such that the approximation of Y with $aX + b$ is the best possible for the quadratic risk, i.e. minimizes
$$\mathbb{E}[(Y - aX - b)^2]\,.$$

Exercise 6. Let X be a standard gaussian rv independent of ϵ, a Rademacher rv such that $\mathbb{P}(\epsilon = \pm 1) = 1/2$.

1. Determine the distributions of $X\epsilon$ and $(X, Y = X\epsilon)$.
2. Deduce the distribution of Y given X.
3. Check that $\mathbb{E}[Y|X] = 0$ but Y is dependent of X. Conclude that (X, Y) cannot be gaussian.