Stochastic Simulation and Monte Carlo Methods: Exercise Class 2.

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Exercise 1. The truncated normal distribution is the one of $N \sim \mathcal{N}(0,1)$ given that $N > 0$.

- Determine the density of this distribution,
- Compute the efficiency M of the reject sampling from the proposal $Y \sim \mathcal{E}xp(1)$.

Exercise 2. The Beta distribution is given thanks to its density

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad x \in [0,1]\,.$$

Give a reject sampling of the distribution Beta(2,5) with proposal density $Y \sim \mathcal{U}(0,1)$.

Exercise 3. Give the constants a, b_- and b_+ for the ratio algorithm fo the following densities h (up to constants)

- Cauchy $h(x) = \frac{1}{1+x^2}$,
- Exponential $h(x) = e^{-x}$, $x > 0$,
- Gaussian $h(x) = e^{-x^2/2}$,

and for each case compute the efficiency of the associated rejection step.

Exercise 4. Let (X, Y) be a standard gaussian vector in $\mathbb{R}^2$. Show that its radial part $R = X^2 + Y^2$ and its angular part $\Theta = \arctan(Y/X)$ are independent and distributed as $\mathcal{E}xp(1/2)$ and $\mathcal{U}nif(0, 2\pi)$.

Exercise 5. Show that the mixing distribution such that $X \mid Y \sim \mathcal{P}oisson(Y)$ with mixing variable $Y \sim \mathcal{G}amma(n, \lambda)$ with $n \geq 1$, $\lambda > 0$ is a negative binomial distribution of the form

$$\mathbb{P}(X = k) = \binom{k+n-1}{k} p^n (1-p)^k \qquad k \geq 0\,.$$

Determine p in function of λ and interpret X as the number of trials necessary for obtaining n successes in a specific experiment.

Exercise 6. Assume that $Z \sim \mathcal{B}]\nabla\backslash(p)$, $p \in (0,1)$, $X|Z = 0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$ and $X|Z = 1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ for μ_1, $\mu_2 \in \mathbb{R}$ and σ_1^2, $\sigma_2^2 > 0$.

1. What is the name of the distribution of X? Is it discrete or continuous?
2. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[X]$. Calculate the parameter $p \in (0,1)$ given μ_1, μ_2 and $\mathbb{E}[X]$.
3. Compute $\mathrm{Var}\,(X)$ and comment.