Stochastic Modelisation: TD1.

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Exercise 1. Prove that if $\mathbb{E}[Y^2] < \infty$ then

$$\mathrm{Var}\,(Y) = \mathrm{Var}\,(\mathbb{E}[Y \mid X]) + \mathbb{E}[\mathrm{Var}\,(Y \mid X)]$$

where $\mathrm{Var}\,(Y \mid X) = \mathbb{E}[Y^2 \mid X] - \mathbb{E}[Y \mid X]^2$.

Exercise 2. Assume that $(X, Y) \sim \mathcal{N}_2(0, \Sigma)$ with $\mu = (\mu_1, \mu_2)^T$ and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. What is the distribution of Y given X?

Exercise 3. Let $0 < p < 1$ and $U, U_1, \ldots, U_n \sim \mathcal{U}(0, 1)$ iid. Determine the distribution of $X = \mathbb{1}_{\{U \le p\}}$ and $Y = \sum_{i=1}^n \mathbb{1}_{\{U_i \le p\}}$.

Exercise 4.

- Let $X \sim \mathcal{E}xp(\lambda)$, calculate $\mathbb{P}(k - 1 < X \le k)$.
- Show that $X = -\log(U)/\lambda$ in distribution, where $U \sim \mathcal{U}(0, 1)$.
- We recall that $Y \sim \mathcal{G}eo(p)$ iff $\mathbb{P}(Y = r) = p(1 - p)^{k-1}$. Show that $Y = \lceil \log(U)/\log(1 - p) \rceil$ in distribution.