Time Series Analysis: TD2.

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Exercise 1. We have the following theorem

Theorem 1 (Hannan)**.** *If (X_t) satisfies an ARMA(p,q) model $\phi(L)X_t = \gamma(L)Z_t$ with $\theta = (\phi_1, \dots, \phi_p, \gamma_1, \dots, \gamma_q)^\top \in \mathcal{C}$ and (Z_t) SWN(σ^2), $\sigma^2 > 0$, then the QMLE is asymptotically normal*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d.} \mathcal{N}_{p+q}\left(0, Var(AR_p, \dots, AR_1, MA_q, \dots, MA_1)^{-1}\right)$$

where (AR_t) and (MA_t) are the stationary AR(p) and AR(q) satisfying

$$\phi(L)AR_t = \eta_t, \qquad \gamma(L)MA_t = \eta_t, \qquad t \in \mathbb{Z},$$

with the same SWN(1) (η_t).

Let us consider the ARMA(1,1) model

$$X_t = \phi X_{t-1} + Z_t + \gamma Z_{t-1},$$

with $\theta = (\phi, \gamma)^\top$.

1. Determine the causal representation of the time series (AR_t) and (MA_t) under the appropriate condition on θ.
2. Compute the matrix of variance-covariance $\mathrm{Var}(AR_1, MA_1)$.
3. Determine the assumption for inverting the matrix and interpret it.
4. Invert the matrix.
5. Deduce a consistent estimator of the asymptotic variance by plugging in the QMLE $\hat{\theta}_n$.
6. Deduce a confidence region of asymptotic level $1 - \alpha$ for the coefficient θ.

Exercise 2. Consider the GARCH(1,1) model

$$\begin{cases} Z_t = \sigma_t W_t, \qquad t \in \mathbb{Z}, \\ \sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha Z_{t-1}^2, \end{cases}$$

with $\omega > 0$, $\alpha, \beta \geq 0$ and $(W_t) \in \mathrm{SWN}(1)$.

1. Show that if $\alpha + \beta \geq 1$, there exists no second order stationary solution.
2. Let $0 \leq \alpha + \beta < 1$ and $1 - \kappa\alpha^2 - \beta^2 - 2\alpha\beta > 0$, with $\kappa = \mathbb{E}\left[Z_t^4\right]$. Show that (σ_t^2) admits a second order stationary solution and determine the kurtosis $\frac{\mathbb{E}[X_t^4]}{\mathrm{Var}\,[X_t]^2}$.
3. Show that if $1 - \kappa\alpha^2 - \beta^2 - 2\alpha\beta \leq 0$, then (σ_t^2) has no second order stationary solution.