Time Series Analysis: TD3.

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Exercise 1. Let us recall the canonical state-space representation of a causal ARMA model

$$\begin{cases} X_t = (1,0,\dots,0)\mathbf{Y}_t + Z_t, & \text{Space equation,} \\ \mathbf{Y}_t = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ \phi_k & \cdots & \phi_2 & \phi_1 \end{pmatrix} \mathbf{Y}_{t-1} + \begin{pmatrix} \psi_1 \\ \vdots \\ \vdots \\ \psi_k \end{pmatrix} Z_{t-1}, & \text{State equation,} \end{cases}$$

where $k = \max(p,q)$, $\phi_j = 0$ for $j \geq p$ and ψ_j are the first coefficients of the polynomial $\psi(z) = \phi^{-1}(z)\gamma(z)$.

1. Show that the coefficients ψ_j satisfy the recursion $\psi_i = \gamma_i + \sum_{j=0}^{i-1} \phi_{i-j}\psi_j$ starting from $\psi_0 = 1$.
2. Denoting $G = (1,0,\dots,0)^\top$, F the matrix in the state equation and $H = (\psi_1,\dots,\psi_k)$, show that $G^\top F^{i-1} H = \psi_i$ for all $1 \leq i \leq k$.
3. Show that the characteristic polynomial satisfies $\det(zI_k - F) = z^k - \phi_1 z^{k-1} - \dots - \phi_k$. Applying the Cayley-Hamilton theorem, we get $F^k - \phi_1 F^{k-1} - \dots - \phi_k I_k = 0$ (admitted).
4. Using the state and the space equations, show that $X_{t+i} = G^\top F^i \mathbf{Y}_t + G^\top \sum_{j=1}^{i} F^{j-1} H Z_{t+i-j} + Z_{t+i}$, $i \geq 0$.
5. Deduce from previous questions 2-3-4 that

$$X_{t+k} - \phi_1 X_{t-k-1} - \cdots - \phi_k X_t = (-\phi_k, \dots, -\phi_1, 1) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \psi_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \psi_k & \cdots & \psi_1 & 1 \end{pmatrix} \begin{pmatrix} Z_t \\ \vdots \\ \vdots \\ Z_{t+k} \end{pmatrix}.$$

6. Conclude thanks to question 1.

Exercise 2. Consider the dynamical model for constant coefficient

$$\begin{cases} X_{t+1} = \theta_t^\top \mathbf{X}_t + Z_{t+1} & \text{Space equation,} \\ \theta_{t+1} = \theta_t & \text{State equation.} \end{cases}$$

We apply the Kalman filter under the gaussian assumption (Z_t) iid $\mathcal{N}(0,1)$.

1. We denote $\hat{\theta}_n = \Pi_n(\theta_n)$ and $\hat{X}_{n+1} = \hat{\theta}_n^\top \mathbf{X}_n$. Show that

$$\hat{\theta}_{n+1} = \hat{\theta}_n + \frac{1}{r_{n+1}^L} \Omega_n \mathbf{X}_n (X_{n+1} - \hat{X}_{n+1})$$

   where $\Omega_n = \mathbb{E}[(\hat{\theta}_n - \theta_n)(\hat{\theta}_n - \theta_n)^T]$ and $r_{n+1}^L = \mathbb{E}[(X_{n+1} - \hat{X}_{n+1})^2]$.

2. Show that

$$\Omega_{n+1} = \Omega_n - \frac{1}{1 + \mathbf{X}_n^\top \Omega_n \mathbf{X}_n} \Omega_n \mathbf{X}_n \mathbf{X}_n^\top \Omega_n. \qquad n \geq 0.$$

3. Let $\mathbf{A}$ a $k \times k$ matrix and $\mathbf{X}$ a vector of $\mathbb{R}^k$. Check that

$$(\mathbf{A} + \mathbf{X}\mathbf{X}^\top)^{-1} = \mathbf{A}^{-1} - \frac{1}{1 + \mathbf{X}^\top \mathbf{A}^{-1} \mathbf{X}} \mathbf{A}^{-1} \mathbf{X}\mathbf{X}^\top \mathbf{A}^{-1}.$$

4. Deduce by recursion that $\Omega_{n+1} = \left(\sum_{t=1}^n \mathbf{X}_t \mathbf{X}_t^\top \right)^{-1}$.

5. Identify $\Omega_{n+1}\mathbf{X}_n$ as ${r_{n+1}^L}^{-1} \Omega_n \mathbf{X}_n$ and the Kalman filter as an Online Newton step algorithm for quadratic losses $\ell_t(\theta) = (X_{t+1} - \theta^\top \mathbf{X}_t)^2$ and its specific learning rate η.