Time Series Analysis: TD5.

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Exercise 1. Consider the GARCH(1,1) model

$$\begin{cases} Z_t = \sigma_t W_t, & t \in \mathbb{Z}, \\ \sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha Z_{t-1}^2, \end{cases}$$

with $\omega > 0$, $\alpha, \beta \geq 0$ and $(W_t) \in \mathrm{SWN}(1)$.

1. Show that if $\alpha + \beta \geq 1$, there is no second order stationary solution.
2. Let $0 \leq \alpha + \beta < 1$ $\kappa\alpha^2 + \beta^2 + 2\alpha\beta < 1$, with $\kappa = \mathbb{E}\left[W_t^4\right]$. Show that (σ_t^2) admits a second order stationary solution and determine the kurtosis $\frac{\mathbb{E}[X_t^4]}{\mathrm{Var}\,[X_t]^2}$.
3. Show that if $1 - \kappa\alpha^2 - \beta^2 - 2\alpha\beta \leq 0$, then (σ_t^2) has no second order stationary solution.

Exercise 2. Let us recall the canonical state-space representation of a causal ARMA model

$$\begin{cases} X_t = (1, 0, \ldots, 0)\mathbf{Y}_t + Z_t, & \text{Space equation,} \\ \mathbf{Y}_t = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \\ \phi_k & \cdots & \phi_2 & \phi_1 \end{pmatrix} \mathbf{Y}_{t-1} + \begin{pmatrix} \psi_1 \\ \vdots \\ \vdots \\ \psi_k \end{pmatrix} Z_{t-1}, & \text{State equation,} \end{cases}$$

where $r = \max(p, q)$, $\phi_j = 0$ for $j > p$ and ψ_j are the first coefficients of the polynomial $\psi(z) = \phi^{-1}(z)\gamma(z)$.

1. Denoting $\gamma_j = 0$ for $j > q$, show that the coefficients ψ_j satisfy the recursion $\psi_i = \gamma_i + \sum_{j=0}^{i-1} \phi_{i-j}\psi_j$ starting from $\psi_0 = 1$.
2. Denoting $G = (1, 0, \ldots, 0)^\top$, F the matrix in the state equation and $H = (\psi_1, \ldots, \psi_k)$, show that $G^\top F^{i-1} H = \psi_i$ for all $1 \leq i \leq r$.
3. Show that the characteristic polynomial satisfies $\det(zI_r - F) = z^r - \phi_1 z^{r-1} - \ldots - \phi_r$. Applying the Cayley-Hamilton theorem, we get $F^r - \phi_1 F^{r-1} - \ldots - \phi_r I_r = 0$ (admitted).
4. Using the state and the space equations, show that $X_{t+i} = G^\top F^i \mathbf{Y}_t + G^\top \sum_{j=0}^{i-1} F^{i-1-j} H Z_{t+j} + Z_{t+i}$, $i \geq 0$.

5. Deduce from previous questions 2-3-4 that

$$X_{t+r} - \phi_1 X_{t+r-1} - \cdots - \phi_r X_t = (-\phi_r, \ldots, -\phi_1, 1) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \psi_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \psi_r & \cdots & \psi_1 & 1 \end{pmatrix} \begin{pmatrix} Z_t \\ \vdots \\ \vdots \\ Z_{t+r} \end{pmatrix}.$$

6. Conclude thanks to question 1.