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AN INVARIANCE PRINCIPLE FOR WEAKLY DEPENDENT STATIONARY GENERAL MODELS

BY

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Abstract. The aim of this article is to refine a weak invariance principle for stationary sequences given by Doukhan & Louhichi (1999). Since our conditions are not causal our assumptions need to be stronger than the mixing and causal θ-weak dependence assumptions used in Dedecker & Doukhan (2003). Here, if moments of order > 2 exist, a weak invariance principle and convergence rates in the CLT are obtained; Doukhan & Louhichi (1999) assumed the existence of moments with order > 4. Besides the previously used η- and κ-weak dependence conditions, we introduce a weaker one, λ, which fits the Bernoulli shifts with dependent inputs.

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1. INTRODUCTION

Let $(X_t)_{t\in\mathbb{Z}}$ be a real-valued stationary process. A huge amount of applications make use of such times series.

Several ways of modeling weak dependence have already been proposed. One of the most popular is the notion of mixing, see [9] for bibliography; this notion leads to a very nice asymptotic theory, in particular a weak invariance principle under very sharp conditions (see [26] for the strong mixing case). Such mixing conditions entail restrictions on the model. For example, Andrews exhibits in [1] the simple counter-example of an auto-regressive process which does not satisfy any mixing condition and innovations need much regularity in both MA(∞) and Markov models. Doukhan & Louhichi introduced in [7] new weak dependence conditions in order to solve those problems. We intend to sharpen their assumptions leading to a weak invariance principle.

A common approach to derive a weak invariance principle for stationary sequences is based on a martingale difference approximation. This approach was first explored by Gordin in [14]; necessary and sufficient conditions were found by

Heyde in [15]. Let $\mathcal{M}_t$ be a filtration. Heyde's martingale difference approximation is equivalent to the existence of moments of order 2 and

$$\sum_{t=0}^{\infty}\mathbb{E}(X_t|\mathcal{M}_0) \quad \text{and} \quad \sum_{t=0}^{\infty}(X_{-t}-\mathbb{E}(X_{-t}|\mathcal{M}_0)) \qquad \text{converge in } \mathbb{L}^2. \tag{1.1}$$

Martingale theory leads directly to invariance principles (see also [27]). In the following, the adapted case refers to the special case where X_t is $\mathcal{M}_t$-measurable. The natural filtration is written as $\mathcal{M}_t = \sigma(Y_i, i \leqslant t)$ for independent and identically distributed inputs $(Y)_{t\in\mathbb{Z}}$; thus X_t can be written as a function of the past inputs:

$$X_t = H(Y_t, Y_{t-1}, \ldots). \tag{1.2}$$

Then only the first series in (1.1) needs to be considered. Using the Lindeberg technique, Dedecker & Rio relax (1.1) in [6]. Bernstein's blocks method allowed Peligrad & Utev to also improve on (1.1) in [22]. Such projective conditions are related to dependence coefficients; Dedecker & Doukhan obtain sharp results for the causal θ-dependence in [5] and Merlevède *et al.* address the mixing cases in a nice survey paper [20].

Martingale difference approximation is not always easy, for instance in the particular case where a natural filtration does not exist. The most striking example is given by associated sequences $(X_t)_{t\in\mathbb{Z}}$. Let us recall this notion. A series is said to be associated if $\text{Cov}(f_1, f_2) \geqslant 0$ for any two coordinatewise nondecreasing functions f_1 and f_2 of $(X_{t_1}, \ldots, X_{t_m})$ with $\text{Var}(f_1) + \text{Var}(f_2) < \infty$. However, Newman & Wright obtain in [21] a weak invariance principle under the existence of second order moments and

$$\sigma^2 = \sum_{t\in\mathbb{Z}} \text{Cov}(X_0, X_t) < \infty. \tag{1.3}$$

Theorems 2.1 and 2.2 propose invariance principles under general assumptions: they apply to the non-causal Bernoulli shifts with weakly dependent inputs $(Y_t)_{t\in\mathbb{Z}}$,

$$X_t = H(Y_{t-j}, j \in \mathbb{Z}). \tag{1.4}$$

Heredity of weak dependence through such non-linear functionals follows from a new λ-weak dependence property; a function of a λ-weak dependence process is λ-weakly dependent, see Section 3.2. Analogous models with dependent inputs are already considered by [3]. If $X_t = \sum_{j\in\mathbb{Z}} \alpha_j Y_{t-j}$, Peligrad & Utev prove in [23] that the Donsker invariance principle holds for X as soon as it holds for the innovation process Y. The non-linearity of H considered here is an important feature which has not been frequently discussed in the past. The condition of moments with order > 4 on the observations needed in [7] is reduced to a one of moments with order > 2 and the results rely on specific decays of the dependence coefficients. We do not reach the second order moment condition of [15] (or projective conditions) and [21]. We *conjecture* that some times series satisfy weak dependence conditions with fast enough decay rates in order to ensure a Donsker type theorem but they do not satisfy neither condition (1.1) nor other projective criterion (see [20]) nor association nor Gaussianity. More general models (1.4) are considered here while causal models (1.2) fit to the adapted case and to projective conditions. However, proving this conjecture is really difficult since condition (1.1) has to be checked for each σ-algebra $\mathcal{M}_0$.

The paper is organized as follows. In Section 2 we introduce various weak dependent coefficients in order to state our main results. Section 3 is devoted to examples of weak dependent models for which we discuss our results. We shall focus on examples of λ-weakly dependent sequences. Proofs are given in the last section; we first derive conditions ensuring the convergence of the series σ^2. A bound of the Δ-moment of a sum (with $2 < \Delta < m$) is proved in Section 4.2; this bound is of an independent interest since *eg.* it directly yields the strong laws of large number. The standard Lindeberg method with Bernstein's blocks is developed in Section 4.3 and yields our versions of the Donsker theorem. Convergence rates of the CLT are obtained in Section 4.4.

2. DEFINITIONS AND MAIN RESULTS

2.1. Weak dependence assumptions.

DEFINITION 2.1 (Doukhan & Louhichi, 1999). The process $(X_t)_{t\in\mathbb{Z}}$ is said to be (ε,ψ)-weakly dependent if there exist a sequence $\varepsilon(r)\downarrow 0$ (as $r\uparrow\infty$) and a function $\psi:\mathbb{N}^2\times(\mathbb{R}^+)^2\to\mathbb{R}^+$ such that

$$|\mathrm{Cov}(f(X_{s_1},\ldots,X_{s_u}),g(X_{t_1},\ldots,X_{t_v})|\leqslant\psi(u,v,\mathrm{Lip}\,f,\mathrm{Lip}\,g)\varepsilon(r),$$

for any $r\geqslant 0$ and any $(u+v)$-tuples such that $s_1\leqslant\cdots\leqslant s_u\leqslant s_u+r\leqslant t_1\leqslant\cdots\leqslant t_v$, where the real valued functions f,g are defined respectively on $\mathbb{R}^u$ and $\mathbb{R}^v$, satisfy $\|f\|_\infty\leqslant 1$, $\|g\|_\infty\leqslant 1$ and are such that $\mathrm{Lip}\,f+\mathrm{Lip}\,g<\infty$ where

$$\mathrm{Lip}\,f=\sup_{(x_1,\ldots,x_u)\neq(y_1,\ldots,y_u)}\frac{|f(x_1,\ldots,x_u)-f(y_1,\ldots,y_u)|}{|x_1-y_1|+\cdots+|x_u-y_u|}$$

Specific functions ψ yield notions of weak dependence appropriate to describe various examples of models:

- κ-weak dependence for which $\psi(u,v,a,b)=uvab$, in this case we simply denote $\varepsilon(r)$ as $\kappa(r)$;

- κ' (causal) weak dependence for which $\psi(u,v,a,b) = vab$, in this case we simply denote $\varepsilon(r)$ as $\kappa'(r)$; this is the causal counterpart of κ coefficients which is recalled only for completeness;

- η-weak dependence, $\psi(u,v,a,b) = ua + vb$, in this case we write $\varepsilon(r) = \eta(r)$ for short;

- θ-weak dependence is a causal dependence which refers to the function $\psi(u,v,a,b) = vb$, in this case we simply denote $\varepsilon(r) = \theta(r)$ (see [5]); this is the causal counterpart of η coefficients which is recalled only for completeness;

- λ-weak dependence $\psi(u,v,a,b) = uvab + ua + vb$, in this case we write $\varepsilon(r) = \lambda(r)$.

Remark 2.1. Besides the fact that it includes η and κ-weak dependences, this new notion of λ-weak dependence will be proved to be convenient, for example, for the Bernoulli shifts with associated inputs (see Lemma 2.1 below).

Remark 2.2. If functions f and g are complex-valued, the previous inequalities remain true if we substitute $\varepsilon(r)/2$ to $\varepsilon(r)$. A useful case of such complex-valued functions is $f(x_1,\dots,x_u) = \exp(it(x_1+\cdots+x_u))$ for each $t\in\mathbb{R}$, $u\in\mathbb{N}^*$ and $(x_1,\dots,x_u)\in\mathbb{R}^u$ (see Section 4.3). This indeed corresponds to the characteristic function adapted to derive the convergence in distribution.

2.2. Main results. Let $(X_t)_{t\in\mathbb{Z}}$ be a real-valued stationary sequence of mean 0 satisfying

$$\mathbb{E}|X_0|^m < \infty, \qquad \text{for a real number } m > 2. \tag{2.1}$$

Let us assume that

$$\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0,X_k) = \sum_{k\in\mathbb{Z}} \mathbb{E}X_0X_k \geqslant 0. \tag{2.2}$$

Denote by W the standard Brownian motion and by W_n the partial sums process

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i, \quad \text{for } t\in[0,1],\ n\geqslant 1. \tag{2.3}$$

We now present our main results, which are new versions of the Donsker weak invariance principle.

Theorem 2.1 (κ-dependence). *Assume that the 0-mean κ-weakly dependent stationary process $(X_t)_{t\in\mathbb{Z}}$ satisfies eqn. (2.1) and $\kappa(r) = O(r^{-\kappa})$ (as $r\uparrow\infty$) for $\kappa > 2 + \frac{1}{m-2}$ then the previous expression σ^2 is finite and*

$$W_n(t) \xrightarrow[n\to\infty]{\mathrm{D}} \sigma W(t), \quad \text{in the Skorohod space } D([0,1]).$$

Remark 2.3. Under the more restrictive κ' condition, Bulinski & Sashkin obtain invariance principles with the sharper assumption $\kappa' > 1 + \frac{1}{m-2}$ in [4]. Our loss is explained by the fact that κ'-weakly dependent sequences satisfy $\kappa'(r) \geqslant \sum_{s \geqslant r} \kappa_s$. This simple bound directly follows from the definitions.

The following result relaxes the previous dependence assumptions at the price of a faster decay of the dependence coefficients.

Theorem 2.2 (λ-dependence). *Assume that the 0-mean λ-weakly dependent stationary inputs satisfies eqn. (2.1) and $\lambda(r) = O(r^{-\lambda})$ (as $r \uparrow \infty$) for $\lambda > 4 + \frac{2}{m-2}$ then σ^2 is finite and*

$$W_n(t) \xrightarrow[n\to\infty]{\mathrm{D}} \sigma W(t), \quad \textit{in the Skorohod space } D([0,1]).$$

Remark 2.4. We do not achieve better results for η or θ-weak dependence cases than the one for $\lambda-$dependence. In comparison with the result obtained by [5], our results are not as good under θ-weak dependence. We work under more restrictive moment conditions than these authors. The same remark applies for all projective measures of dependence; here, we refer to results by [15], [21], [6] and [22].

Remark 2.5. However, the example of Section 3.2 stresses the fact that such results are not systematically better than those of Theorem 2.2; for such general examples, we even conjecture that theorems of [15], [21], [6] or [22] do not apply.

Remark 2.6. The technique of the proofs is based on the Lindeberg method. In fact, we prove that $|\mathbb{E}\left(\phi(S_n/\sqrt{n}) - \phi(\sigma N)\right)| = o\left(n^{-c}\right)$ (ϕ denotes here the characteristic function) for $0 < c < c^*$ where c^* depends only on the parameters m and κ or λ respectively. If m and κ (or λ) both tend to infinity, we notice that $c^* \to \frac{1}{4}$. As κ or λ tends to infinity and $m < 3$, c^* always remains smaller than $(m-2)/(2m-2)$ (see Proposition 4.2 in Section 4.4 for more details).

Remark 2.7. Using a smoothing lemma also yields an analogous bound for the uniform distance

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{1}{\sqrt{n}} S_n \leqslant x\right) - \mathbb{P}\left(\sigma N \leqslant x\right) \right| = o\left(n^{-c}\right), \qquad \text{for some } c < c'.$$

A first and easy way to control c' is to let $c' = c^*/4$ but the corresponding rate is a really bad one (see *e.g.* in [8]). The Essen inequality holds with the optimal exponent $\frac{1}{2}$ in the independent and identically distributed case (see [24]) and [26] reaches the exponent $\frac{1}{3}$ in the case of strongly mixing sequences. In Proposition 4.2 of Section 4.4, we achieve $c' > c^*/4$. Analogous results have been settled in [10] for weakly dependent random fields. Previous results by [15], [21], [6] or [22] do not derive such convergence rates for the Kolmogorov distance.

Let us denote by $\mathbb{R}^{(\mathbb{Z})} = \bigcup_{I>0} \{z \in \mathbb{R}^{\mathbb{Z}} / \ z_i = 0, \ |i| > I\}$, the set of finite sequences of real numbers. We consider functions $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$ such that if $x, y \in \mathbb{R}^{(\mathbb{Z})}$ coincide for all indexes but one, let say $s \in \mathbb{Z}$, then

$$|H(x) - H(y)| \leqslant b_s(\|z\|^{\ell} \vee 1)|x_s - y_s| \tag{2.4}$$

where $z \in \mathbb{R}^{(\mathbb{Z})}$ is defined by $z_s = 0$ and $z_i = x_i = y_i$ if $i \neq s$. Here $\|x\| = \sup_{i\in\mathbb{Z}} |x_i|$. In Section 3.2, we prove the existence of $X_n = \lim_{I\to\infty} H\big((Y_{n-j}\mathbb{1}_{\{j\leqslant I\}})_{j\in\mathbb{Z}}\big)$ where $(Y_t)_{t\in\mathbb{Z}}$ is a weakly dependent real-valued input process. We denote this process by $X_n = H(Y_{n-j}, j \in \mathbb{Z})$ for simplicity and we derive its λ-weak invariance properties. Various asymptotic results follow, among which our weak invariance principle, Theorem 2.2.

Corollary 2.1. *Let $(Y_t)_{t\in\mathbb{Z}}$ be a stationary λ-weakly dependent process (with dependence coefficients $\lambda_Y(r)$) and $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$ satisfying the condition given by (2.4) for some $\ell \geqslant 0$. Let us assume that there exist real numbers m, m' with $\mathbb{E}|Y_0|^{m'} < \infty$ such that $m > 2$ and $m' \geqslant (\ell+1)m$.*

Then $X_n = H(Y_{n-i}, i \in \mathbb{Z})$ exists and satisfies the weak invariance principle in the following cases:

- ***Geometric case:** $b_r \leqslant Ce^{-b|r|}$ and $\lambda_Y(r) \leqslant De^{-ar}$ for $a, b, C, D > 0$.*
- ***Riemannian case:** If $b_r \leqslant C(1+|r|)^{-b}$ for some $b > 2$ and $\lambda_Y(r) \leqslant Dr^{-a}$ for $a, C, D > 0$ with*

$$(2.5) \quad a > \frac{1+b}{b-1}\left(4 + \frac{2}{m-2}\right), \ \text{if } \ell = 0, \ b > 1;$$
$$a > \frac{b(m'-1+\ell)}{(b-2)(m'-1-\ell)}\left(4 + \frac{2}{m-2}\right), \ \text{if } \ell > 0, \ b > 2.$$

Remark 2.8. The previous conditions are also tractable in the mixed cases. We explicitly state them for $\ell > 0$:

- $b_r \leqslant Ce^{-b|r|}$, $\lambda_Y(r) \leqslant Dr^{-a}$, if moreover $a > \dfrac{m'-1+\ell}{m'-1-\ell}\left(4 + \dfrac{2}{m-2}\right)$ and $b, C, D > 0$.
- $b_r \leqslant C|r|^{-b}$ and $\lambda_Y(r) \leqslant De^{-ar}$, for $a, C, D > 0$ with $b > \dfrac{6m-10}{m-2}$.

3. EXAMPLES

Theorem 2.1 is useful to derive the weak invariance principle in various cases. This section is aimed at a detailed treatment of the Bernoulli shifts with dependent inputs. The important class of Lipschitz functions of dependent inputs is presented in a separate section. The importance of our results is highlighted by the models of the first subsection. More general non-linear models are considered in the second subsection. Some of those examples illustrate the conjecture we made in the introduction but we were not able to formally prove it.

3.1. Lipschitz processes with dependent inputs. Consider Lipschitz functions $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$, *i.e.* such that eqn. (2.4) is satisfied for $\ell = 0$. A simple example of this situation is the two-sided linear sequence

$$X_t = \sum_{i \in \mathbb{Z}} \alpha_i Y_{t-i} \tag{3.1}$$

with dependent inputs $(Y_t)_{t\in\mathbb{Z}}$. As quoted by [17] for the case of linear processes with dependent input there exists a very general solution; essentially any Donsker type theorem for the stationary inputs implies the central limit theorem for any linear process driven by such inputs. More precisely, Theorem 5 of [23] states that this process even satisfies the Donsker invariance principle if $\sum_j |\alpha_j| < \infty$.

A simple example of Lipschitz non-linear functional of dependent inputs is

$$X_t = \left| \sum_{i \in \mathbb{Z}} \alpha_i Y_{t-i} \right| - \mathbb{E} \left| \sum_{i \in \mathbb{Z}} \alpha_i Y_{-i} \right| \tag{3.2}$$

In this case the inequality (2.4) holds with $\ell = 0$ and $b_r \leqslant |\alpha_r|$.

Another example of this situation is the following stationary process

$$X_t = Y_t \left(a + \sum_{j \neq 0} a_j Y_{t-j} \right) - \mathbb{E} Y_t \left(a + \sum_{j \neq 0} a_j Y_{t-j} \right),$$

where the inputs $(Y_t)_{t\in\mathbb{Z}}$ are bounded. In this case, the inequality (2.4) also holds with $\ell = 0$ and $b_s \leqslant 2\|Y_0\|_\infty |a_s|$.

To apply our result, we compute the weak dependence coefficients of such models.

Lemma 3.1. *Let $(Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary process with a finite moment of order $m \geqslant 1$ and $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$ satisfying the condition (2.4) for $\ell = 0$ and some nonnegative sequence $(b_s)_{s\in\mathbb{Z}}$ such that $L = \sum_j b_j < \infty$. Then,*

- *the process $X_n = H(Y_{n-j}, j \in \mathbb{Z}) := \lim_{I\to\infty} H\left(Y_{n-j} \mathbb{1}_{\{j \leqslant I\}}, j \in \mathbb{Z}\right)$ is a strictly stationary process with finite moments of order m.*
- *if the input process $(Y_t)_{t\in\mathbb{Z}}$ is λ-weakly dependent (the weak dependence coefficients are denoted by $\lambda_Y(r)$), then $(X_t)_{t\in\mathbb{Z}}$ is λ-weakly dependent with*

$$\lambda(k) = \inf_{2r \leqslant k} \left[2 \sum_{|i| \geqslant r} b_i \|Y_0\|_1 + (2r+1)^2 L^2 \lambda_Y(k-2r) \right].$$

- *if the input process* $(Y_t)_{t\in\mathbb{Z}}$ *is* η*-weakly dependent (the weak dependence coefficients are denoted by* $\eta_Y(r)$*) then* $(X_t)_{t\in\mathbb{Z}}$ *is* η*-weakly dependent and*

$$\eta(k) = \inf_{2r\leqslant k}\left[2\sum_{|i|\geqslant r} b_i\|Y_0\|_1 + (2r+1)L\eta_Y(k-2r)\right].$$

REMARK 3.1. Let $(Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary process with a finite moment of order $m > 2$. If $L = \sum_j |\alpha_j| < \infty$, the process $X_n = \sum_{j\in\mathbb{Z}}\alpha_j Y_{n-j}$ is a strictly stationary process with finite moments of order m which satisfies the assumptions of Lemma 3.1 with $b_j = |\alpha_j|$. Even if the weak invariance principle is already given in [23] our result is of an independent interest, *e.g.* for functional estimation purposes. For non-linear Lipschitz functionals it yields new central limit theorems.

The result of Theorems 2.1 and 2.2 holds systematically in geometric cases. Then it is assumed Riemannian decays, *i.e.* there exists $\alpha, C > 0$ such that

$$b_r \leqslant Cr^{-\alpha}.$$

The conditions from [15] are compared below with the conditions of Theorems 2.1 and 2.2 for specific classes of inputs $(Y_t)_{t\in\mathbb{Z}}$.

3.1.1. LARCH(∞) inputs. A vast literature is devoted to the study of conditionally heteroskedastic models. A simple equation in terms of a vector-valued process allows a unified treatment of those models, see [12]. Let $(\xi_t)_{t\in\mathbb{Z}}$ be an independent and identically distributed centered real-valued sequence and $a, a_j, j\in\mathbb{N}^*$ be real numbers. LARCH(∞) models are solutions of the recurrence equation

$$Y_t = \xi_t\left(a + \sum_{j=1}^{\infty} a_j Y_{t-j}\right). \tag{3.3}$$

We provide below sufficient conditions for the following chaotic expansion

$$Y_t = \xi_t\left(a + \sum_{k=1}^{\infty}\sum_{j_1,\dots,j_k\geqslant 1} a_{j_1}\xi_{t-j_1}a_{j_2}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}a\right). \tag{3.4}$$

Assume that $\Lambda = \|\xi_0\|_m\sum_{j\geqslant 1}|a_j| < 1$ then one (essentially unique) stationary solution of eqn. (3.3) in $\mathbb{L}^m$ is given by eqn. (3.4). This solution is θ-weakly dependent with $\theta_Y(r) \leqslant Kr^{1-a}\log^{a-1} r$ for some constant $K > 0$. This implies the same bound on their coefficients $(\lambda_Y(r))_{r\geqslant 0}$. Condition (2.5) gives the weak invariant principle for $(X_t)_{t\in\mathbb{Z}}$ under the conditions $\mathbb{E}|\xi_0|^m < +\infty$ for $m > 2$, $\alpha > 1$, and

$$a > \frac{1+\alpha}{\alpha-1}\left(4 + \frac{2}{m-2}\right) + 1.$$

The model (3.1) is also an Heyde's martingale difference approximation (1.1) as soon as

$$\sum_{k\geqslant 1}\sqrt{\sum_{i\geqslant k}\alpha_i^2}<+\infty.$$

Necessary conditions for weak invariance principle follow as $\alpha > 3/2$, $|a_j| \leqslant Cj^{-a}$ for some $a > 1$, $\mathbb{E}\xi_0^2 < +\infty$, and $\|\xi_0\|_2 \sum_{j\geqslant 1}|a_j| < 1$. These conditions are not optimal since in this case the process is adapted to the filtration $\mathcal{M}_t = \sigma(\xi_i, i \leqslant t)$. Peligrad & Utev extend in [22] the Donsker theorem to the cases where $\alpha > 1/2$. Thus, our conditions are not optimal compared to those of [23] in the linear case as in eqn. (3.1). However, for non-linear Lipschitz functional, the result seems to be new.

3.1.2. Non-causal LARCH(∞) inputs. The previous approach extends for the case of non-causal LARCH(∞) inputs

$$Y_t = \xi_t\left(a + \sum_{j\neq 0} a_j Y_{t-j}\right).$$

Doukhan *et al.* prove in [12] the same results of existence as for the previous causal case (just replace summation over $j > 0$ by summation over $j \neq 0$) and the dependence becomes of the η type with

$$\eta(r) = \left(\|\xi_0\|_\infty \sum_{0\leqslant 2k<r} k\Lambda^{k-1}A\left(\frac{r}{2k}\right) + \frac{\Lambda^{r/2}}{1-\Lambda}\right)\mathbb{E}|\xi_0||a|$$

where $A(x) = \sum_{|j|\geqslant x}|a_j|$, $\Lambda = \|\xi_0\|_\infty \sum_{j\geqslant 1}|a_j| < 1$. ¿From condition (2.5) the weak invariance principle holds for $(X_t)_{t\in\mathbb{Z}}$ if $\|\xi_0\|_\infty < \infty$, $\alpha > 1$ with

$$a > \frac{1+\alpha}{\alpha-1}\left(4 + \frac{2}{m-2}\right) + 1.$$

Notice that a very restrictive new assumption is that inputs need to be uniformly bounded in this non-causal case. This result is new, a conjecture is that (1.1) does not hold.

3.1.3. Non-causal, non-linear inputs. The weak dependence properties of non-causal and non-linear inputs Y_t are recalled, see [7] for more details. Let $H : (\mathbb{R}^d)^{\mathbb{Z}} \to \mathbb{R}$ be a measurable function. If the sequence $(\xi_n)_{n\in\mathbb{Z}}$ is independent and identically distributed on $\mathbb{R}^d$, the Bernoulli shift with input process $(\xi_n)_{n\in\mathbb{Z}}$ is defined as

$$Y_n = H\left((\xi_{n-i})_{i\in\mathbb{Z}}\right), \qquad n \in \mathbb{Z}.$$

Such Bernoulli's shifts are η-weakly dependent (see [7]) with $\eta(r) \leqslant 2\delta_{[r/2]}$ if

$$\mathbb{E}\left|H\left(\xi_j, j\in\mathbb{Z}\right) - H\left(\xi_j \mathbb{1}_{|j|\leqslant r}, j\in\mathbb{Z}\right)\right| \leqslant \delta_r. \tag{3.5}$$

Then condition (2.5) leads to the invariance principle for $(X_t)_{t\in\mathbb{Z}}$ if $\mathbb{E}|Y_0|^m < \infty$ for $m > 2$, $\alpha > 1$ and $\delta_r \leqslant K r^{-\delta}$ for

$$\delta > \frac{1+\alpha}{\alpha-1}\left(4+\frac{2}{m-2}\right).$$

Conditions (1.1) of [15] do not give clear conditions on coefficients for these models. We do not know other weak invariance principle in that general context.

3.1.4. Associated inputs. A process is associated if $\mathrm{Cov}\left(f(Y^{(n)}), g(Y^{(n)})\right) \geqslant 0$ for any coordinatewise non-decreasing function $f, g : \mathbb{R}^n \to \mathbb{R}$ such that the previous covariance makes sense with $Y^{(n)} = (Y_1, \ldots, Y_n)$. The κ-weak dependence condition is known to hold for associated or Gaussian sequences. In both cases this condition will hold with

$$\kappa(r) = \sup_{j \geqslant r} |\,\mathrm{Cov}(Y_0, Y_j)|$$

Notice the absolute values are needed only in the second case since for associated processes these covariances are nonnegative. Independent sequences as well are associated and Pitt proves in [25] that a Gaussian process with nonnegative covariances is also associated. Finally, we recall that non-decreasing functions of associated sequences remain associated. Associated models are classically built this way from independent and identically distributed sequences, see [18].

Suppose that the inputs $(Y_t)_{t\in\mathbb{Z}}$ are such that $\kappa(r) \leqslant C r^{-a}$ (for some $a, C > 0$). For the associated cases and model (3.1), the invariance principle of [21] follows from remark of [19] as soon as $\mathbb{E}Y^2 < +\infty$, $a > 1$ and $\alpha > 1$. These conditions are optimal, they correspond to $\sum_j \mathrm{Cov}(X_0, X_j) < \infty$. Such strong conditions are due to the fact that zero correlation implies independence for associated processes. Our conditions for invariance principle are much stronger: $\mathbb{E}|Y|^m < +\infty$ with $m > 2$, $\alpha > 1$ and

$$a > \frac{1+\alpha}{\alpha-1}\left(4+\frac{2}{m-2}\right).$$

For non-linear Lipschitz cases as in eqn. (3.2) the result seems to be new. In the special case of κ-weak dependent inputs that are not associated, the optimal weak invariance principle of [21] does not apply, see [7] for examples.

3.2. The Bernoulli shifts with dependent inputs. Let $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$ be a measurable (non necessarily Lipschitz) function and $X_n = H(Y_{n-i}, i \in \mathbb{Z})$. Such models are proved to exhibit either λ or η-weak dependence properties. Because the Bernoulli shifts of κ-weak dependent inputs are neither κ nor η-weakly dependent, the κ case is here included in the λ one.

Consider the non-Lipschitz function H defined by

$$H(x) = \sum_{k=0}^{K} \sum_{j_1,\dots,j_k} a^{(k)}_{j_1,\dots,j_k} x_{j_1} \cdots x_{j_k}.$$

In this case, Lemma 3.1 does not apply. To derive weak dependence properties of such processes, we assume that H satisfies the condition (2.4) with $\ell \neq 0$, which remains a stronger assumption than for the case of independent inputs, see eqn. (3.5). Relaxing Lipschitz assumption on H is possible if we assume the existence of higher moments for the inputs. The following lemma gives both the existence and the weak dependence properties of such models

LEMMA 3.2. *Let $(Y_i)_{i\in\mathbb{Z}}$ be a stationary process and $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$ satisfies condition (2.4) for some $\ell > 0$ and some sequence $b_j \geqslant 0$ such that $\sum_j |j| b_j < \infty$. Let us assume that there exist a pair of real numbers (m, m') with $\mathbb{E}|Y_0|^{m'} < \infty$ such that $m \geqslant 1$ and $m' \geqslant (\ell+1)m$. Then,*

- *the process $X_n = H(Y_{n-i}, i \in \mathbb{Z})$ is well defined in $\mathbb{L}^m$: it is a strictly stationary process;*
- *if the input process $(Y_i)_{i\in\mathbb{Z}}$ is λ-weakly dependent (the weak dependence coefficients are denoted by $\lambda_Y(r)$), then X_n is λ-weakly dependent and there exists a constant $c > 0$ such that*

$$\lambda(k) = c \inf_{r \leqslant [k/2]} \left[\sum_{|j| \geqslant r} |j| b_j + (2r+1)^2 \lambda_Y(k-2r)^{\frac{m'-1-\ell}{m'-1+\ell}} \right];$$

- *if the input process $(Y_i)_{i\in\mathbb{Z}}$ is η-weakly dependent (the weak dependence coefficients are denoted by $\eta_Y(r)$) then X_n is η-weakly dependent and there exists a constant $c > 0$ such that*

$$\eta(k) = c \inf_{r \leqslant [k/2]} \left[\sum_{|j| \geqslant r} |j| b_j + (2r+1)^{1+\frac{\ell}{m'-1}} \eta_Y(k-2r)^{\frac{m'-1-\ell}{m'-1}} \right].$$

Such models where already mentioned in the mixing case by [2] and [3]. The proofs are deferred to Section 4.6.

3.2.1. Volterra models with dependent inputs. Consider the function H defined by

$$H(x) = \sum_{k=0}^{K} \sum_{j_1,\dots,j_k} a^{(k)}_{j_1,\dots,j_k} x_{j_1} \cdots x_{j_k},$$

then if x, y are as in eqn. (2.4)

$$H(x)-H(y)=\sum_{k=1}^{K}\sum_{u=1}^{k}\sum_{\substack{j_1,\ldots,j_{u-1}\\ j_{u+1},\ldots,j_k}} a^{(k)}_{j_1,\ldots,j_{u-1},s,j_{u+1},\ldots,j_k}\times x_{j_1}\cdots x_{j_{u-1}}(x_s-y_s)x_{j_{u+1}}\cdots x_{j_k}.$$

From the triangular inequality we thus derive that the previous lemma 3.2 may be written with $\ell = K-1$,

$$b_s=\sum_{k=1}^{K}\sum\nolimits^{(k,s)}|a^{(k)}_{j_1,\ldots,j_k}|$$

where $\sum^{(k,s)}$ stands for the sums over all indices in $\mathbb{Z}^k$ where one of the indices $j_1,\ldots,j_k$ takes on the value s and

$$L\equiv\sum_{k=0}^{K}\sum_{j_1,\ldots,j_k}\left|a^{(k)}_{j_1,\ldots,j_k}\right|.$$

For example, $|a^{(k)}_{j_1,\ldots,j_k}|\leqslant C\,(j_1\vee\cdots\vee j_k)^{-\alpha}$ or $\leqslant C\exp(-\alpha(j_1\vee\cdots\vee j_k))$ respectively yield $b_s\leqslant C's^{d-1-\alpha}$ or $b_s\leqslant C'e^{-\alpha s}$ for some constant $C'>0$.

3.2.2. Markov stationary inputs. Markov stationary sequences satisfy a recurrence equation

$$Z_n=F(Z_{n-1},\ldots,Z_{n-d},\xi_n)$$

where (ξ_n) is a sequence of independent and identically distributed random variables. In this case $Y_n=(Z_n,\ldots,Z_{n-d+1})$ is a Markov chain $Y_n=M(Y_{n-1},\xi_n)$ with

$$M(x_1,\ldots,x_d,\xi)=(F(x_1,\ldots,x_d,\xi),x_1,\ldots,x_{d-1}). \tag{3.6}$$

Theorem 1.IV.24 of [13] proves that eqn. (3.6) has a stationary solution $(Z_n)_{n\in\mathbb{Z}}$ in $\mathbb{L}^m$ for $m\geqslant 1$ as soon as $\|F(0,\xi)\|_m<\infty$ and there exist a norm $\|\cdot\|$ on $\mathbb{R}^d$ and a real number $a\in[0,1[$ such that $\|F(x,\xi)-F(y,\xi)\|_m\leqslant a\|x-y\|$. In this setting θ-dependence holds with $\theta_Z(r)=O(a^{r/d})$ (as $r\uparrow\infty$). We shall not give more details about the significative examples provided in [11]. Indeed, we already mentioned that our results are sub-optimal in such causal cases; such dependent sequences may however also be used as inputs for the Bernoulli shifts.

3.2.3. Explicit dependence rates. We now specify the decay rates from Lemma 3.2. For standard decays of the previous sequences, it is easy to get the following explicit bounds. Here $b,c,C,D,\lambda,\eta>0$ are constants which may differ from one case to the other.

- If $b_j \leqslant C(|j|+1)^{-b}$ and $\lambda_Y(j) \leqslant Dj^{-\lambda}$, resp. $\eta_Y(j) \leqslant Dj^{-\eta}$ then from a simple calculation, we optimize both terms in order to prove that $\lambda(k) \leqslant ck^{-\lambda\left(1-\frac{2}{b}\right)\frac{m'-1-\ell}{m'-1+\ell}}$, resp. $\eta(k) \leqslant ck^{-\eta\frac{(b-2)(m'-2)}{(b-1)(m'-1)-\ell}}$.
Note that in the case where $m'=\infty$ this exponent may be arbitrarily close to λ for large values of $b>0$. This exponent may thus take all possible values between 0 and λ.

- If $b_j \leqslant Ce^{-|j|b}$ and $\lambda_Y(j) \leqslant De^{-j\lambda}$, respectively $\eta_Y(j) \leqslant De^{-j\eta}$, we obtain $\lambda(k) \leqslant ck^2 e^{-\lambda k\frac{b(m'-1-\ell)}{b(m'-1+\ell)+2\eta(m'-1-\ell)}}$, resp. $\eta(k) \leqslant ck^{\frac{m'-1-\ell}{m'-1}} e^{-\eta k\frac{b(m'-2)}{b(m'-1)+2\eta(m'-2)}}$.
The geometric decay of both $(b_j)_{j\in\mathbb{Z}}$ and the weak dependence coefficients of the inputs ensure the geometric decay of the weak dependence coefficients of the Bernoulli shift.

- If we assume that the coefficients $(b_j)_{j\in\mathbb{Z}}$ associated with the Bernoulli shift have a geometric decay, say $b_j \leqslant Ce^{-|j|b}$ and that $\lambda_Y(j) \leqslant Dj^{-\lambda}$ (resp. $\eta_Y(j) \leqslant Dj^{-\eta}$) we obtain the bounds $\lambda(k) \leqslant ck^{-\lambda\frac{m'-1-l}{m'-1+\ell}}\log^2 k$, resp. $\eta(k) \leqslant ck^{-\eta\frac{m'-2}{m'-1}}\log^{1+\frac{\ell}{m'-1}} k$.
If $m'=\infty$ tightness is reduced by a factor $\log^2 k$ with respect to the dependence coefficients of the input dependent series $(Y_t)_{t\in\mathbb{Z}}$.

- If we assume that the coefficients $(b_j)_{j\in\mathbb{Z}}$ associated with the Bernoulli shift have a Riemannian decay, say $b_j \leqslant C(|j|+1)^{-b}$ and that $\lambda_Y(j) \leqslant De^{-j\lambda}$ (resp. $\eta_Y(j) \leqslant De^{-j\eta}$) we find $\lambda(k) \leqslant ck^{2-b}$, resp. $\eta(k) \leqslant ck^{2-b}$.

All models or functions of models we present here are λ-weakly dependent. We treat some basic examples in detail when a discussion with other results is possible. We believe that for some models, λ-weak invariance properties follow from easy computations, and then, statistical results like our weak invariance principle.

4. PROOFS OF THE MAIN RESULTS

Our proof for central limit theorems is based on a truncation method. For a truncation level $T \geqslant 1$ we shall denote $\overline{X}_k = f_T(X_k) - \mathbb{E}f_T(X_k)$ with $f_T(X) = X \vee (-T) \wedge T$. ¿From now on, we shall use the convenient notation $a_n \preceq b_n$ for two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ when there exists some constant $C>0$ such that $|a_n| \leqslant Cb_n$ for each integer n. We also remark that $\overline{X}_k$ has moments of all orders because it is bounded. In the entire sequel, we denote $\mu = \mathbb{E}|X_0|^m$. For any $a \leqslant m$, we control the moment $\mathbb{E}|f_T(X_0) - X_0|^a$ with Markov inequality

$$\mathbb{E}|f_T(X_0) - X_0|^a \leqslant \mathbb{E}|X_0|^a \mathbb{1}_{\{|X_0|\geqslant T\}} \leqslant \mu T^{a-m},$$

thus using Jensen inequality yields

$$\|\overline{X}_0 - X_0\|_a \leqslant 2\mu^{\frac{1}{a}} T^{1-\frac{m}{a}}. \tag{4.1}$$

Starting from this truncation, we are now able to control the limiting variance as well as the higher order moments.

In this section we prove that the central limit theorems corresponding to the convergence $W_n(1) \to W(1)$ in both Theorems 2.1 and 2.2 hold and we shall provide convergence rates corresponding to these central limit theorems. The weak invariance principle is obtained in a standard way from such central limit theorems and tightness, which follows from Lemma 3.2, by using the classical Kolmogorov-Centsov tightness criterion, see [2]. In the last subsection, we prove Lemma 4.2 that states the properties of our (new) Bernoulli's shifts with dependent inputs.

4.1. Variances.

LEMMA 4.1 (Variances). *If one of the following conditions holds*

$$\sum_{k=0}^{\infty} \kappa(k) < \infty \tag{4.2}$$

$$\sum_{k=0}^{\infty} \lambda(k)^{\frac{m-2}{m-1}} < \infty \tag{4.3}$$

then the series σ^2 is convergent.

PROOF. Using the fact that $\overline{X}_0 = g_T(X_0)$ is a function of X_0 with $\mathrm{Lip}\, g_T = 1$ snd $\|g_T\|_\infty \leqslant 2T$, we derive

$$|\mathrm{Cov}(\overline{X}_0, \overline{X}_k)| \leqslant \kappa(k) \text{ or } (4T+1)\lambda(k), \text{ respectively.} \tag{4.4}$$

In the κ dependent case, the truncation may thus be omitted and

$$|\mathrm{Cov}(X_0, X_k)| \leqslant \kappa(k). \tag{4.5}$$

In the following, we shall only consider λ dependence. We develop

$$\mathrm{Cov}(X_0, X_k) = \mathrm{Cov}(\overline{X}_0, \overline{X}_k) + \mathrm{Cov}(X_0 - \overline{X}_0, X_k) + \mathrm{Cov}(\overline{X}_0, X_k - \overline{X}_k).$$

We use a truncation T (to be determined) and the two previous bounds eqn. (4.1) and eqn. (4.4); then the Hölder inequality with the exponents $1/a + 1/m = 1$ yields

$$\begin{aligned} |\mathrm{Cov}(X_0, X_k)| &\leqslant (4T+1)\lambda(k) + 2\|X_0\|_m \|\overline{X}_0 - X_0\|_a \\ &\leqslant (4T+1)\lambda(k) + 4\mu^{1/a+1/m} T^{1-m/a} \\ &\leqslant (4T+1)\lambda(k) + 4\mu T^{2-m}. \end{aligned}$$

Chosing $T^{m-1} = \mu/\lambda(k)$ we obtain

$$|\mathrm{Cov}(X_0, X_k)| \leqslant 9\mu^{\frac{1}{m-1}} \lambda(k)^{\frac{m-2}{m-1}}. \tag{4.6}$$

■

4.2. A Δ-order moment bound.

LEMMA 4.2. *Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary and centered process. Let us assume that $\mathbb{E}|X_0|^m < \infty$, and that this process is either κ-weakly dependent with $\kappa(r) = O(r^{-\kappa})$ or λ-weakly dependent with $\lambda(r) = O(r^{-\lambda})$. If $\kappa > 2 + \frac{1}{m-2}$, or $\lambda > 4 + \frac{2}{m-2}$, then for all $\Delta > 2$ small enough there exists a constant $C > 0$ such that*

$$\|S_n\|_\Delta \leqslant C\sqrt{n}.$$

REMARK 4.1. $\Delta \in]2, 2+A\wedge B\wedge 1[$ where A and B are constants smaller than $m-2$ and depend on m and respectively κ or λ. Equations (4.10) and (4.11) precise the previous involved constants A and B.

REMARK 4.2. The constant satisfies $C > \left(\frac{5}{2^{(\Delta-2)/2}-1}\right)^{1/\Delta} \sum_{k\in\mathbb{Z}} |\operatorname{Cov}(X_0,X_k)|$. Under the conditions of this lemma, Lemma 4.1 entails

$$c \equiv \sum_{k\in\mathbb{Z}} |\operatorname{Cov}(X_0,X_k)| < \infty.$$

REMARK 4.3. The result is sketched from [4]. However, their dependence condition is of a causal nature while our is not. It explains a loss with respect to the exponents λ and κ. In their κ′-weak dependence setting the best possible value of the exponent is 1 while it is 2 for our non-causal dependence.

Proof of Lemma 4.2. For convenience, let denote in the sequel $\Delta = 2+\delta$ and $m = 2+\zeta$. Like in [16] or [4], we proceed by induction on k for $n \leqslant 2^k$ to prove that

$$\|1+|S_n|\|_\Delta \leqslant C\sqrt{n}. \tag{4.7}$$

We assume that eqn. (4.7) is satisfied for all $n \leqslant 2^{K-1}$. Setting $N = 2^K$ we have to find a bound for $\|1+|S_N|\|_\Delta$. We can divide the sum S_N into three blocks: the first two blocks have the same size $n \leqslant 2^{K-1}$ and are denoted by Q and R; the third block V, located between Q and R, has cardinality $q < n$. We then have $\|1+|S_N|\|_\Delta \leqslant \|1+|Q|+|R|\|_\Delta + \|V\|_\Delta$. The term $\|V\|_\Delta$ is directly bounded by $\|1+|V|\|_\Delta \leqslant C\sqrt{q}$ from the recurrence assumption. Writing $q = N^b$ with $b < 1$, then this term is of order strictly smaller than $\sqrt{N}$. For $\|1+|Q|+|R|\|_\Delta$, we have

$$\begin{aligned}
\mathbb{E}(1+|Q|+|R|)^\Delta &\leqslant \mathbb{E}(1+|Q|+|R|)^2(1+|Q|+|R|)^\delta, \\
&\leqslant \mathbb{E}(1+2|Q|+2|R|+(|Q|+|R|)^2)(1+|Q|+|R|)^\delta.
\end{aligned}$$

We expand the right-hand side of this expression; the following terms appear

- $\mathbb{E}(1+|Q|+|R|)^\delta \leqslant 1+|Q|_2^\delta+|R|_2^\delta \leqslant 1+2c^\delta(\sqrt{n})^\delta$,

- $\mathbb{E}|Q|(1+|Q|+|R|)^\delta \leqslant \mathbb{E}|Q|((1+|R|)^\delta+|Q|^\delta)$
$\leqslant \mathbb{E}|Q|(1+|R|)^\delta+\mathbb{E}|Q|^{1+\delta}$.

  The term $\mathbb{E}|Q|^{1+\delta}$ is bounded by $\|Q\|_2^{1+\delta}$ and then by $c^{1+\delta}(\sqrt{n})^{1+\delta}$. The term $\mathbb{E}|Q|(1+|R|)^\delta$ is bounded by $\|Q\|_{1+\delta/2}\|1+|R|\|_\Delta^\delta$ using Hölder inequality. It is at least of order $cC^\delta(\sqrt{n})^{1+\delta}$, analogous to the latter one, where we exchange the roles of Q and R.

- $\mathbb{E}(|Q|+|R|)^2(1+|Q|+|R|)^\delta$. For this term, we use an inequality from [4]

$$\mathbb{E}(|Q|+|R|)^2(1+|Q|+|R|)^\delta \leqslant \mathbb{E}|Q|^\Delta+\mathbb{E}|R|^\Delta+5(\mathbb{E}Q^2(1+|R|)^\delta+\mathbb{E}R^2(1+|Q|)^\delta).$$

  Now $\mathbb{E}|Q|^\Delta \leqslant C^\Delta(\sqrt{n})^\Delta$ is bounded by using eqn. (4.7). The second term is its analogous with R substituted to Q. The third term has to be handled with a particular care, as follows.

We use the weak dependence notion to control $\mathbb{E}Q^2(1+|R|)^\delta$ and $\mathbb{E}R^2(1+|Q|)^\delta$. Denote by $\overline{X}$ the variable $X\vee T\wedge(-T)$ for a real $T>0$ to be determined later. By extension $\overline{Q}$ and $\overline{R}$ denote the truncated sums of the variables X_i. We have

$$\mathbb{E}|Q|^2(1+|R|)^\delta \leqslant \mathbb{E}Q^2||R|-|\overline{R}||^\delta+\mathbb{E}(Q^2-\overline{Q}^2)(1+|\overline{R}|)^\delta+\mathbb{E}\overline{Q}^2(1+|\overline{R}|)^\delta.$$

We begin with a control of $\mathbb{E}Q^2||R|-|\overline{R}||^\delta$. Using the Hölder inequality with $2/m+1/m'=1$ yields

$$\mathbb{E}Q^2||R|-|\overline{R}||^\delta \leqslant \|Q\|_m^2\|||R|-|\overline{R}||^\delta\|_{m'}$$

$\|Q\|_\Delta$ is bounded using eqn. (4.7) and

$$||R|-|\overline{R}||^{\delta m'} \leqslant |R|^{\delta m'}\mathbb{1}_{\{|R|>T\}} \leqslant |R|^{\delta m'}\mathbb{1}_{|R|>T}.$$

We then bound $\mathbb{1}_{|R|>T} \leqslant (|R|/T)^\alpha$ with $\alpha=m-\delta m'$, hence

$$\mathbb{E}||R|-|\overline{R}||^{\delta m'} \leqslant E|R|^m T^{\delta m'-m}.$$

By convexity and stationarity, we have $\mathbb{E}|R|^m \leqslant n^m\mathbb{E}|X_0|^m$, so that

$$\mathbb{E}Q^2(|R|-|\overline{R}|)^\delta \preceq n^{2+m/m'}T^{\delta-m/m'}.$$

Finally, remarking that $m/m'=m-2$, we obtain

$$\mathbb{E}Q^2(|R|-|\overline{R}|)^\delta \preceq n^m T^{\Delta-m}.$$

We get the same bound for the second term

$$\mathbb{E}(Q^2-\overline{Q}^2)(1+|\overline{R}|)^\delta \preceq n^m T^{\Delta-m}.$$

For the third one, we introduce a covariance term

$$\mathbb{E}\overline{Q}^2(1+|\overline{R}|)^\delta \leqslant \mathrm{Cov}(\overline{Q}^2,(1+|\overline{R}|)^\delta)+\mathbb{E}\overline{Q}^2\mathbb{E}(1+|\overline{R}|)^\delta.$$

The latter is bounded with $|Q|_2^2|R|_2^\delta \leqslant c^\Delta \sqrt{n}^\Delta$. The covariance is controlled as follows by using weak-dependence

- in the κ-dependent case: $n^2T\kappa(q)$,
- in the λ-dependent case: $n^3T^2\lambda(q)$.

We then choose either the truncation $T^{m-\delta-1}=n^{m-2}/\kappa(q)$ or $T^{m-\delta}=n^{m-3}/\lambda(q)$. At this point, the three terms of the decomposition are of the same order

$$\begin{aligned}\mathbb{E}|Q|^2(1+|R|)^\delta &\preceq \left(n^{3m-2\Delta}\kappa(q)^{m-\Delta}\right)^{1/(m-\delta-1)}, \text{ under } \kappa\text{-dependence},\\ \mathbb{E}|Q|^2(1+|R|)^\delta &\preceq \left(n^{5m-3\Delta}\lambda(q)^{m-\Delta}\right)^{1/(m-\delta)}, \text{ under } \lambda\text{-dependence}.\end{aligned}$$

Let $q=N^b$, we note that $n\leqslant N/2$ and this term is of order $N^{\frac{3m-2\Delta+b\kappa(\Delta-m)}{m-\delta-1}}$ under κ-weak dependence and the order $N^{\frac{5m-3\Delta+b\lambda(\Delta-m)}{m-\delta}}$ under λ-weak dependence. Those terms are thus negligible with respect to $N^{\Delta/2}$ if

$$\kappa > \frac{3m-2\Delta-\Delta/2(m-\delta-1)}{b(m-\Delta)}, \text{ under } \kappa\text{-dependence}, \tag{4.8}$$

$$\lambda > \frac{5m-3\Delta-\Delta/2(m-\delta)}{b(m-\Delta)}, \text{ under } \lambda\text{-dependence}. \tag{4.9}$$

Finally, using this assumption, $b<1$ and $n\leqslant N/2$, we derive the bound for some suitable constants $a_1,a_2>0$

$$\mathbb{E}(1+|S_N|)^\Delta \leqslant \left(2^{-\delta/2}C^\Delta+5\cdot 2^{-\delta/2}c^\Delta+a_1N^{-a_2}\right)\left(\sqrt{N}\right)^\Delta.$$

Using the relation between C and c, we conclude that eqn. (4.7) is also true at the step N if the constant C satisfies $2^{-\delta/2}C^\Delta+5\cdot 2^{-\delta/2}c^\Delta+a_1N^{-a_2}\leqslant C^\Delta$. Choose $C>\left(\frac{5c^\Delta+a_1 2^{\delta/2}}{2^{\delta/2}-1}\right)^{1/\Delta}$ with $c=\sum_{k\in\mathbb{Z}}|\mathrm{Cov}(X_0,X_k)|$, then the previous relation holds. Finally, we use eqn. (4.8) and eqn. (4.9) to find a condition on δ.
In the case of κ-weak dependence, we rewrite inequality eqn. (4.8) as

$$0>\delta^2+\delta(2\kappa-3-\zeta)-\kappa\zeta+2\zeta+1.$$

It leads to the following condition on δ

$$\delta < \frac{\sqrt{(2\kappa-3-\zeta)^2+4(\kappa\zeta-2\zeta-1)}+\zeta+3-2\kappa}{2} \wedge 1 = A. \tag{4.10}$$

We do the same in the case of the λ-weak dependence

$$\delta < \frac{\sqrt{(2\lambda-6-\zeta)^2+4(\lambda\zeta-4\zeta-2)}+\zeta+6-2\lambda}{2} \wedge 1 = B. \quad \square \tag{4.11}$$

REMARK 4.4. The bounds A and B are always smaller than ζ.

4.3. Proofs of Theorems 2.1 and 2.2. Let $S = \frac{1}{\sqrt{n}} S_n$ and consider $p = p(n)$ and $q = q(n)$ in such a way that

$$\lim_{n\to\infty} \frac{1}{q(n)} = \lim_{n\to\infty} \frac{q(n)}{p(n)} = \lim_{n\to\infty} \frac{p(n)}{n} = 0$$

and $k = k(n) = n/[p(n)+q(n)]$

$$Z = \frac{1}{\sqrt{n}} (U_1 + \cdots + U_k), \qquad \text{with } U_j = \sum_{i \in B_j} X_i$$

where $B_j =](p+q)(j-1), (p+q)(j-1)+p] \cap \mathbb{N}$ is a subset of p successive integers from $\{1,\ldots,n\}$ such that, for $j \neq j'$, B_j and $B_{j'}$ are at least distant of $q = q(n)$ from each other. We denote by B'_j the block between B_j and B_{j+1} and $V_j = \sum_{i\in B'_j} X_i$. V_k is the last block of X_i between the end of B_k and n. Furthermore, let $\sigma_p^2 = \mathrm{Var}(U_1)/p = \sum_{|i|<p}(1-|i|/p)\mathbb{E}X_0X_i$, and let

$$Y = \frac{U'_1 + \cdots + U'_k}{\sqrt{n}}, \qquad U'_j \sim \mathcal{N}(0, p\sigma_p^2)$$

where the Gaussian variables V_j are mutually independent and also independent of the sequence $(X_n)_{n\in\mathbb{Z}}$. We also consider a sequence $U_1^*, \ldots, U_k^*$ of mutually independent random variables with the same distribution as U_1 and we let $Z^* = \left(U_1^* + \cdots + U_k^*\right)/\sqrt{n}$. In the entire section, we fix $t \in \mathbb{R}$ and we define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = \exp\{itx\}$. Then $\mathbb{E}f(S) - f(\sigma N) = \mathbb{E}f(S) - f(Z) + \mathbb{E}f(Z) - f(Z^*) + \mathbb{E}f(Z^*) - f(Y) + \mathbb{E}f(Y) - f(\sigma N)$. Lindeberg method is devoted to prove that this expression converges to 0 as $n \to \infty$. The first and the last terms in this inequality are referred to as the auxiliary terms in this Bernstein-Lindeberg method. They come from the replacement of the individual initial – non-Gaussian and Gaussian respectively – random variables by their block counterparts. The second term is analogous to that obtained with decoupling and turns the proof of the central limit theorem to the independent case. The third term is referred to as the main term and following the proof under independence it will be bounded above by using

a Taylor expansion. Because of the dependence structure, in the corresponding bounds, some additional covariance terms will appear.
The following subsections are organized as follows: we first consider the auxiliary terms and the main terms are then decomposed by the usual Lindeberg method and the corresponding terms coming from the dependence or the usual remainder terms (standard for the independent case) are considered in separate subsections. In the last one, we collect these calculations to obtain the central limit theorem.

4.3.1. Auxiliary terms. Using Taylor expansions up to the second order, we obtain

$$|\mathbb{E}f(S)-f(Z)| \leqslant \|f'\|_\infty \mathbb{E}|S-Z|$$
$$\text{and} \quad |\mathbb{E}f(Y)-f(\sigma N)| \leqslant \frac{\|f''\|_\infty^2}{2}\mathbb{E}|Y-\sigma N|^2.$$

We note that $Z-S=(V_1+\cdots+V_k)/\sqrt{n}$ is a sum of X_i's for which the number of terms is $\leqslant (k+1)q+p$. Then eqn. (4.6) and eqn. (4.5), under conditions (4.3) or (4.2) respectively, entail:

$$(\mathbb{E}|Z-S|)^2 \leqslant \mathbb{E}|Z-S|^2 \preceq \big((k+1)q+p\big)/n.$$

Now $Y \sim \sqrt{\frac{kp}{n}}\sigma_p N$, thus

$$\mathbb{E}|Y-\sigma N|^2 \leqslant \left|\frac{kp}{n}-1\right|\sigma_p^2+|\sigma_p^2-\sigma^2|.$$

Remarking that $|kp/n-1|^2 \preceq \big((k+1)q+p\big)/n$, it remains to bound the quantity

$$|\sigma_p^2-\sigma^2| \quad \leqslant \quad \sum_{|i|<p}\frac{|i|}{p}|\mathbb{E}X_0X_i|+\sum_{|i|>p}|\mathbb{E}X_0X_i|.$$

Let $a_i=|\mathbb{E}X_0X_i|$, under conditions (4.3) or (4.2) (respectively), the series $\sum_{i=0}^{\infty} a_i$ converge thus $s_j=\sum_{i=j}^{\infty}a_i \underset{j\to\infty}{\longrightarrow} 0$ and

$$|\sigma_p^2-\sigma^2| \leqslant 2\sum_{i=0}^{p-1}\frac{i}{p}\cdot a_i+2s_p \leqslant \frac{2}{p}\sum_{i=0}^{p-1}s_i+2s_p.$$

Cesaro lemma entails that term $|\sigma_p^2-\sigma^2|$ converges to 0.
Hence $|\mathbb{E}f(S)-f(Z)|+|\mathbb{E}f(Y)-f(\sigma N)|$ tends to 0 as $n\uparrow\infty$.
To determine the convergence rate, we assume that $a_i=O(i^{-\alpha})$ for some $\alpha>1$; then

$$|\sigma_p^2-\sigma^2| \preceq p^{1-\alpha}.$$

Remarking that $a_i = \mathbb{E}X_0X_i = \mathrm{Cov}(X_0, X_i)$, we then use equations (4.5) and (4.6) and we find $\alpha = \kappa$ or $\alpha = \lambda(m-2)/(m-1)$ depending of the weak-dependence setting. With $p = n^a$, $q = n^b$ for 2 constants a and b and from the relation $\|f^{(j)}\|_\infty \leqslant |t|^j$, those bounds become, up to a constant

$$|t|\left(n^{(b-a)/2} + n^{(a-1)/2}\right) + t^2\left(n^{b-a} + n^{a(1-\kappa)}\right), \text{ in the } \kappa\text{-weak dependence setting,}$$

$$|t|\left(n^{(b-a)/2} + n^{(a-1)/2}\right) + t^2\left(n^{b-a} + n^{a(1-\lambda(m-2)/(m-1))}\right), \text{ for } \lambda\text{-weak dependence.}$$

4.3.2. Main terms. It remains to control the second and the third terms of the sum. They are bounded as usual by

$$|\mathbb{E}f(Z) - f(Z^*)| \leqslant \sum_{j=1}^{k} |\mathbb{E}\Delta_j|, \quad |\mathbb{E}f(Z^*) - f(Y)| \leqslant \sum_{j=1}^{k} |\mathbb{E}\Delta'_j|,$$

where $\Delta_j = f(W_j + x_j) - f(W_j + x_j^*)$, for $j = 1, \ldots, k$ with $x_j = \frac{1}{\sqrt{n}}U_j$, $x_j^* = \frac{1}{\sqrt{n}}U_j^*$, $W_j = w_j + \sum_{i>j} x_i^*$, $w_j = \sum_{i<j} x_i$ and $\Delta'_j = f(W'_j + x_j^*) - f(W'_j + x'_j)$, for $j = 1, \ldots, k$ with $x'_j = \frac{1}{\sqrt{n}}U'_j$, $W'_j = \sum_{i<j} x_i^* + \sum_{i>j} x'_i$.
Exploiting the special form of f and the independence properties of the variables U_i^* and U'_i, we can write

$$\begin{aligned}
\mathbb{E}\Delta_j &= \left(\mathbb{E}f(w_j)f(x_j) - \mathbb{E}f(w_j)\mathbb{E}f(x_j^*)\right)\mathbb{E}f\left(\sum_{i>j} x_i^*\right), \\
\mathbb{E}\Delta'_j &= \left(\mathbb{E}f(x_j^*) - \mathbb{E}f(x'_j)\right)\mathbb{E}f\left(W'_j\right).
\end{aligned}$$

We then control the two terms $\mathbb{E}f\left(\sum_{i>j} x_i^*\right)$ and $\mathbb{E}f\left(W'_j\right)$ by the fact that $\|f\|_\infty \leqslant 1$ and we use the coupling to introduce a covariance term

$$\begin{aligned}
|\mathbb{E}\Delta_j| &\leqslant \left|\mathrm{Cov}\left(f\left(\sum_{i<j} x_i\right), f(x_j)\right)\right|, \\
|\mathbb{E}\Delta'_j| &\leqslant \left|\mathbb{E}f(x_j^*) - \mathbb{E}f(x'_j)\right|.
\end{aligned}$$

- For Δ_j, we use weak dependence.
  To do so, write $|\mathbb{E}\Delta_j| = |\mathrm{Cov}[F(X_m, m \in B_i, i < j), G(X_m, m \in B_j)]|$, with $F(z_1, \ldots, z_{kp}) = f\left(\sum_{i<j} u_i/\sqrt{n}\right)$ where $u_i = \sum_{\ell \in B_i} z_\ell$. We verify that $\|F\|_\infty \leqslant 1$

and we control Lip$F$:

$$\left| f\left(\frac{1}{\sqrt{n}}\sum_{i<j}\sum_{\ell\in B_i} z_\ell\right) - f\left(\frac{1}{\sqrt{n}}\sum_{i<j}\sum_{\ell\in B_i} z'_\ell\right)\right| \leqslant \left|1-\exp it\left(\frac{1}{\sqrt{n}}\sum_{i<j}\sum_{\ell\in B_i}(z_\ell - z'_\ell)\right)\right| \leqslant \frac{|t|}{\sqrt{n}}\sum_{\ell=1}^{kp}|z_\ell - z'_\ell|.$$

Similarly, for $G(z_1,\dots,z_p) = f\left(\sum_{i=1}^{p} z_i/\sqrt{n}\right)$, we have $\|G\|_\infty = 1$ and $\mathrm{Lip}\,G \preceq |t|/\sqrt{n}$. We then distinguish the two cases of weak dependence, remarking the gap between the left and the right terms in the covariance is at least q.

- In the κ-weak dependent setting: $|\mathbb{E}\Delta_j| \preceq kp\cdot p\cdot\frac{|t|}{\sqrt{n}}\cdot\frac{|t|}{\sqrt{n}}\cdot\kappa(q)$.
- Under the λ dependence condition:
  $|\mathbb{E}\Delta_j| \preceq \left(kp\cdot p\cdot\frac{|t|}{\sqrt{n}}\cdot\frac{|t|}{\sqrt{n}} + kp\cdot\frac{|t|}{\sqrt{n}} + p\cdot\frac{|t|}{\sqrt{n}}\right)\cdot\lambda(q)$.

Note that these bounds do not depend on j:

$$\begin{aligned}|\mathbb{E}f(Z)-f(Z^*)| &\preceq kp\cdot t^2\cdot\kappa(q), && \text{under } \kappa,\\ &\preceq kp\cdot(t^2+|t|\sqrt{k/p})\cdot\lambda(q), && \text{under } \lambda.\end{aligned}$$

Knowing that $p=n^a$, $q=n^b$, $\kappa(r)=O(r^{-\kappa})$ or $\lambda(r)=O\left(r^{-\lambda}\right)$, these convergence rates respectively become $n^{1-\kappa b}$ or $n^{1+(1/2-a)_+-\lambda b}$ in the κ or the λ dependence context.

- For Δ'_j, Taylor expansions up to order 2 or 3 respectively give:

$$\begin{aligned}|f(x^*_j)-f(x'_j)| &\leqslant |x^*_j - x'_j|\|f'\|_\infty + \frac{1}{2}(x^*_j-x'_j)^2\|f''\|_\infty + r_j\\ r_j &\leqslant \frac{1}{2}\|f''\|_\infty(x^*_j-x'_j)^2, \text{ or}\\ &\leqslant \frac{1}{6}\|f'''\|_\infty|x^*_j-x'_j|^3,\end{aligned}$$

For an arbitrary $\delta\in[0,1]$, we have:

$$\begin{aligned}\mathbb{E}r_j &\preceq \mathbb{E}(t^2(|x^*_j|^2+|x'_j|^2)\wedge|t|^3(|x^*_j|^3+|x'_j|^3))\\ &\preceq \mathbb{E}(t^2|x^*_j|^2\wedge|t|^3|x^*_j|^3)+\mathbb{E}(t^2|x'_j|^2\wedge|t|^3|x'_j|^3)\\ &\preceq |t|^{2+\delta}\left(\mathbb{E}|x^*_j|^{2+\delta}+\mathbb{E}|x'_j|^{2+\delta}\right).\end{aligned}$$

By the stationarity of the sequence $(X_i)_{i\in\mathbb{Z}}$, we obtain

$$|\mathbb{E}\Delta'_j| \quad \preceq \quad |t|^{2+\delta} n^{-1-\frac{\delta}{2}} \left(\mathbb{E}|S_p|^{2+\delta} \vee p^{1+\frac{\delta}{2}}\right).$$

Lemma 4.2 allows us to fin a bound for $\mathbb{E}|S_p|^{2+\delta}$. If $\kappa > 2+\frac{1}{\zeta}$, or $\lambda > 4+\frac{2}{\zeta}$, where $\kappa(r) = O(r^{-\kappa})$ or $\lambda(r) = O\left(r^{-\lambda}\right)$ then there exist $\delta \in]0, \zeta \wedge 1[$ and $C > 0$ such that

$$\mathbb{E}|S_p|^{2+\delta} \leqslant Cp^{1+\delta/2}.$$

We then obtain

$$|\mathbb{E}f(Z^*) - f(Y)| \preceq |t|^{2+\delta} k(p/n)^{1+\delta/2}.$$

Because $p = n^a$, this bound is of order $n^{(a-1)\delta/2}$ in both κ and λ-weak dependence settings.

We now collect the previous bounds to conclude that a multidimensional CLT holds under assumptions of both Theorems 2.1 and 2.2. Tightness follows from the Kolmogorov-Chentsov criterion (see [2]) and Lemma 4.2; thus both Theorems 2.1 and 2.2 follow from repeated application of the previous CLT. □

4.4. Rates of convergence. Rates of convergence are now presented in two propositions of independent interest. We compute explicit bounds for both the difference of characteristic functions and the Berry-Esséen inequalities.

PROPOSITION 4.1. *Let $(X_t)_{t\in\mathbb{Z}}$ be a weakly dependent stationary process satisfying eqn. (2.1) with $m = 2+\zeta$ then the difference between the characteristic functions is bounded by*

$$\left|\mathbb{E}\left(e^{itS_n/\sqrt{n}} - e^{it\sigma N}\right)\right| = o(n^{-c}),$$

for some $c < c^*$ and all $t \in \mathbb{R}$ where c^* depends of the weak dependent coefficients

- *under κ-weak dependence, if $\kappa(r) = O(r^{-\kappa})$ for $\kappa > 2+\frac{1}{\zeta}$, then we have $c^* = \dfrac{(\kappa-1)A}{A+2\kappa(1+A)}$ where*

$$A = \frac{\sqrt{(2\kappa-3-\zeta)^2+4(\kappa\zeta-2\zeta-1)}+\zeta+3-2\kappa}{2} \wedge 1.$$

- *under λ-weak dependence, if $\lambda(r) = O(r^{-\lambda})$ for $\lambda > 4+\frac{2}{\zeta}$, then we obtain $c^* = \dfrac{(\lambda+1)B}{2+B+2\lambda(1+B)}$ where*

$$B = \frac{\sqrt{(2\lambda-6-\zeta)^2+4(\lambda\zeta-4\zeta-2)}+\zeta+6-2\lambda}{2} \wedge 1,$$

We use the following Essen inequality in Proposition 4.2

THEOREM 4.1 (Theorem 5.1 p.142 of [24]). *Let X and Y be 2 random variables and assume that Y is Gaussian. Let F and G be their distribution functions with corresponding characteristic functions f and g. Then, for every $T > 0$, we have for suitable constants b and c*

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \leqslant b \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right| dt + \frac{c}{T}. \tag{4.12}$$

PROPOSITION 4.2 (A rate in the Berry Essen bounds). *Let $(X_t)_{t\in\mathbb{Z}}$ be a real stationary process satisfying Proposition 4.1 assumptions. We obtain*

$$\sup_x |F_n(x) - \Phi(x)| = o\left(n^{-c}\right)$$

with $c < c'$ where $c' = c^*/(3+A)$ or $c' = c^*/(3+B)$, respectively, in κ or λ-weak dependence contexts (A, B and c^* are defined in Proposition 4.1).

PROOF OF PROPOSITION 4.1. In the previous section, the different terms have already been bounded as follows:

- In the κ-weak dependence case, the exponents of n in the bounds obtained in section 4.3 are
  - for the auxiliary terms: $(b-a)/2$, $(a-1)/2$ and $a(1-\kappa)$,
  - for the main terms: $1-\kappa b$ and $(a-1)\delta/2$.

  Because $\delta < 1$ and $b < a < 1$, we remark that $(a-1)\delta/2 > (a-1)/2$ and $1-\kappa b > a(1-\kappa)$. The only rate of the auxiliary term it remains to consider is $(b-a)/2$ and we obtain

  $$a^* = \frac{2+\delta+2\kappa\delta}{\delta+2\kappa(1+\delta)} \in \left]b^*, \frac{\delta}{1+\delta}\right[, \quad b^* = \frac{2+a^*}{1+2\kappa} \in \left]0, a^*\right[.$$

  We conclude with standard calculations and with the help of the inequality $\delta < A$ (see eqn. (4.10)).

- We have the equivalent in the λ-weak dependence case
  - for the auxiliary terms: $(b-a)/2$, $(a-1)/2$ and $a(1-\lambda)$,
  - for the main terms: $1+(1/2-a)_+ - \lambda b$ and $(a-1)\delta/2$.

  Only three rates give the asymptotic: $(a-1)\delta/2$, $1+(1/2-a)_+ - \lambda b$ and $(b-a)/2$. In the previous case, the optimal choice of a^* was smaller than

1/2. Then we have to consider here the rate $3/2-a-\lambda b$ and not $1-\lambda b$. Thus

$$a^* = \frac{3+\delta+2\lambda\delta}{2+\delta+2\lambda(1+\delta)} \in \left]b^*, \frac{\delta}{1+\delta}\right[,$$
$$b^* = \frac{3+2\delta}{2+\delta+2\lambda(1+\delta)} \in]0, a^*[$$

Finally, we obtain a rate of n^{-c^*} using the inequality eqn. (4.11). ■

PROOF OF PROPOSITION 4.2. Let choose a^* and b^* as in the proof of proposition 4.1. We now need to make precise the impact of t on the different term of the bound of the $\mathbb{L}^1$ distance between the characteristic functions of S and σN. Up to a constant independent of t, the Kolmogorov distance is bounded by $\left(|t|+t^2+|t|^{2+C}\right)n^{-c^*}$. Here $C=A$ or B in the two contexts of dependence. Using Theorem 4.1 for a well chosen value of T, we obtain the result of proposition 4.2. ■

4.5. Proof of Lemma 3.1. The case of Lipschitz functions of dependent inputs is divided in two sections devoted respectively to the definition of such models and to their weak dependence properties.

4.5.1. Existence. Let $Y^{(s)} = (Y_{-i}1\!\!1_{|i|<s})_{i\in\mathbb{Z}}$, $Y_+^{(s)} = (Y_{-i}1\!\!1_{-s<i\leqslant s})_{i\in\mathbb{Z}}$ for $s\in\mathbb{Z}$ and $H\left(Y^{(\infty)}\right) = \lim_{s\to\infty} H\left(Y^{(s)}\right)$. In order to prove the existence of the Bernoulli shift with dependent inputs, we show that X_0 is the sum of a normally convergent series in $\mathbb{L}^m$; formally

$$X_0 = H\big(Y^{(\infty)}\big) = H(0) + \Big(H\big(Y^{(1)}\big) - H(0)\Big) + \sum_{s=1}^{\infty} H\big(Y^{(s+1)}\big) - H\big(Y_+^{(s)}\big) + \Big(H\big(Y_+^{(s)}\big) - H\big(Y^{(s)}\big)\Big).$$

¿From eqn. (2.4) we obtain

$$\begin{aligned}
\left\|H\left(Y^{(1)}\right)-H(0)\right\|_m &\leqslant b_0\|Y_0\|_m,\\
\left\|H\left(Y^{(s+1)}\right)-H\left(Y_+^{(s)}\right)\right\|_m &\leqslant b_{-s}\|Y_{-s}\|_m,\\
\left\|H\left(Y_+^{(s)}\right)-H\left(Y^{(s)}\right)\right\|_m &\leqslant b_s\|Y_s\|_m.
\end{aligned}$$

By $(Y_t)_{t\in\mathbb{Z}}$'s stationarity we get

$$\|X_0\|_m \leqslant \left\|H\left(Y^{(1)}\right)-H(0)\right\|_m + \sum_{s=1}^{\infty}\left\|H\left(Y^{(s+1)}\right)-H\left(Y_+^{(s)}\right)\right\|_m + \left\|H\left(Y_+^{(s)}\right)-H\left(Y^{(s)}\right)\right\|_m \leqslant \sum_{i\in\mathbb{Z}} b_i\|Y_0\|_m \tag{4.13}$$

Analogously, the process $X_t = H(Y_{t-i}, i \in \mathbb{Z})$ is well defined as the sum of a normally convergent series in $\mathbb{L}^m$. The stationarity of $(X_t)_{t\in\mathbb{Z}}$ holds from that of the input process $(Y_t)_{t\in\mathbb{Z}}$.

4.5.2. Weak dependence properties. Let $X_n^{(r)} = H(Y^{(r)})$ and $X_{\mathbf{s}} = (X_{s_1}, \dots, X_{s_u})$, $X_{\mathbf{t}} = (X_{t_1}, \dots, X_{t_v})$ for any $k \geqslant 0$ and any $(u+v)$-tuple such that $s_1 < \cdots < s_u \leqslant s_u + k \leqslant t_1 < \cdots < t_v$. Then we have for all f, g satisfying $\|f\|_\infty, \|g\|_\infty \leqslant 1$ and $\mathrm{Lip}\, f + \mathrm{Lip}\, g < \infty$

$$|\mathrm{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leqslant |\mathrm{Cov}(f(X_{\mathbf{s}}) - f(X_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}))| \tag{4.14}$$

$$+ \ |\mathrm{Cov}(f(X_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}) - g(X_{\mathbf{t}}^{(r)}))| \tag{4.15}$$

$$+ \ |\mathrm{Cov}(f(X_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}^{(r)}))|. \tag{4.16}$$

Using the fact that $\|g\|_\infty \leqslant 1$, we bound the term in eqn. (4.14)

$$2\,\mathrm{Lip}\, f \cdot \mathbb{E}\left|\sum_{i=1}^{u}(X_{s_i} - X_{s_i}^{(r)})\right| \leqslant 2u\,\mathrm{Lip}\, f \max_{1\leqslant i\leqslant u} \mathbb{E}\left|X_{s_i} - X_{s_i}^{(r)}\right|.$$

Applying inequality (4.13) in the case where $m = 1$, we obtain $\mathbb{E}\left|X_{s_i} - X_{s_i}^{(r)}\right| \leqslant \sum_{i\geqslant r} b_i \|Y_0\|_1$. The second term (4.15) is bounded in a similar way.
The last term (4.16) can be written as

$$\left|\mathrm{Cov}(F^{(r)}(Y_{s_i+j}, 1 \leqslant i \leqslant u, |j| \leqslant r), G^{(r)}(Y_{t_i+j}, 1 \leqslant i \leqslant v, |j| \leqslant r)\right|,$$

where $F^{(r)} : \mathbb{R}^{u(2r+1)} \to \mathbb{R}$ and $G^{(r)} : \mathbb{R}^{v(2r+1)} \to \mathbb{R}$. Under the assumption $r \leqslant [k/2]$, we use the $\varepsilon = \eta$ or λ-weak dependence of Y in order to bound this covariance term by $\psi(\mathrm{Lip}\, F^{(r)}, \mathrm{Lip}\, G^{(r)}, u(2r+1), v(2r+1))\varepsilon_{k-2r}$, with respectively $\psi(u,v,a,b) = ua + vb$ or $\psi(u,v,a,b) = uvab + ua + vb$. We compute

$$\mathrm{Lip}\, F^{(r)} = \sup \frac{|f(H(x_{s_i+l}, 1 \leqslant i \leqslant u, |l| \leqslant r) - f(H(y_{s_i+l}, 1 \leqslant i \leqslant u, |l| \leqslant r)|}{\sum_{i=1}^{u} \sum_{-r\leqslant l\leqslant r} |x_{s_i+l} - y_{s_i+l}|},$$

where the sup extends to $x \neq y$ where $x, y \in \mathbb{R}^{u(2r+1)}$. Notice now that if x, y are sequences with $x_i = y_i = 0$ if $|i| \geqslant r$ then repeated applications of the condition (2.4) yields

$$|H(x) - H(y)| \leqslant \sum_{|i|\leqslant r} b_i |x_i - y_i| \leqslant L \sum_{|i|\leqslant r} |x_i - y_i| \tag{4.17}$$

where $L = \sum_{i\in\mathbb{Z}} b_i$. Repeating inequality eqn. (4.17), we obtain

$$|F^{(r)}(x) - F^{(r)}(y)| \leqslant \mathrm{Lip}\, f\, L \sum_{i=1}^{u} \sum_{-r\leqslant l\leqslant r} |x_{s_i+l} - y_{s_i+l}|$$

and we get $\mathrm{Lip}\, F^{(r)} \leqslant \mathrm{Lip}\, f \cdot L$. Similarly $\mathrm{Lip}\, G^{(r)} \leqslant \mathrm{Lip}\, g L$.
Under η-weak dependent inputs, we bound the covariance

$$|\operatorname{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leqslant (u \mathrm{Lip}\, f + v \mathrm{Lip}\, g) \times \left[2 \sum_{|i| \geqslant r} b_i \|Y_0\|_1 + (2r+1) L \eta_Y(k-2r) \right].$$

Under λ-weak dependent inputs

$$|\operatorname{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leqslant (u \mathrm{Lip}\, f + v \mathrm{Lip}\, g + uv \mathrm{Lip}\, f \mathrm{Lip}\, g) \times \times \Big(\Big\{ 2 \sum_{|i| \geqslant r} b_i \|Y_0\|_1 + (2r+1) L \lambda_Y(k-2r) \Big\} \vee (2r+1)^2 L^2 \lambda_Y(k-2r) \Big). \square$$

4.6. Proof of Lemma 3.2.

4.6.1. Existence. We decompose X_0 as above in the case $\ell = 0$. Here, we bound each terms by

$$\begin{aligned} |H(Y^{(1)}) - H(0)| &\leqslant b_0 |Y_0| \\ |H(Y^{(s+1)}) - H(Y_+^{(s)})| &\leqslant b_{-s} (\|Y_+^{(s)}\|_\infty^l \vee 1) |Y_{-s}| \\ |H(Y_+^{(s)}) - H(Y^{(s)})| &\leqslant b_s (\|Y^{(s)}\|_\infty^l \vee 1) |Y_s| \end{aligned}$$

Using Hölder inequality yields

$$\mathbb{E}\left|H(Y^{(1)}) - H(0)\right| + \sum_{s=1}^{\infty} \mathbb{E}\left|H(Y^{(s+1)}) - H(Y_+^{(s)})\right| + \mathbb{E}\left|H(Y_+^{(s)}) - H(Y^{(s)})\right| \leqslant \sum_{i \in \mathbb{Z}} 2|i| b_i (\|Y_0\|_1 + \|Y_0\|_{l+1}^{l+1})$$

Hence assumptions $\ell + 1 \leqslant m'$ and $\sum_{i \in \mathbb{Z}} |i| b_i < \infty$ together imply that the variable $H(Y)$ is well defined in $\mathbb{L}^1$. In the same manner, the process $X_n = H(Y_{n-i}, i \in \mathbb{Z})$ is well defined. The proof extends in $\mathbb{L}^m$ if $m \geqslant 1$ is such that $(\ell+1)m \leqslant m'$.

4.6.2. Weak dependence properties. Here, we exhibit some Lipschitz functions and we then truncate inputs. We write $\overline{Y} = Y \vee (-T) \wedge T$ for a truncation T set below. Denote $X_n^{(r)} = H(Y^{(r)})$ and $\overline{X}_n^{(r)} = H(\overline{Y}^{(r)})$. Furthermore, for any $k \geqslant 0$ and any $(u+v)$-tuple such that $s_1 < \cdots < s_u \leqslant s_u + k \leqslant t_1 < \cdots < t_v$, we set $X_{\mathbf{s}} = (X_{s_1}, \ldots, X_{s_u})$, $X_{\mathbf{t}} = (X_{t_1}, \ldots, X_{t_v})$ and $\overline{X}_{\mathbf{s}}^{(r)} = (\overline{X}_{s_1}^{(r)}, \ldots, \overline{X}_{s_u}^{(r)})$, $\overline{X}_{\mathbf{t}}^{(r)} = (\overline{X}_{t_1}^{(r)}, \ldots, \overline{X}_{t_v}^{(r)})$. Then we have for all f, g satisfying $\|f\|_\infty, \|g\|_\infty \leqslant 1$ and $\mathrm{Lip}\, f + \mathrm{Lip}\, g < \infty$

$$|\operatorname{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leqslant |\operatorname{Cov}(f(X_{\mathbf{s}}) - f(\overline{X}_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}))| \tag{4.18}$$

$$+ \; |\operatorname{Cov}(f(\overline{X}_{\mathbf{s}}^{(r)}), g(X_{\mathbf{t}}) - g(\overline{X}_{\mathbf{t}}^{(r)}))| \tag{4.19}$$

$$+ \; |\operatorname{Cov}(f(\overline{X}_{\mathbf{s}}^{(r)}), g(\overline{X}_{\mathbf{t}}^{(r)}))|. \tag{4.20}$$

Using the fact that $\|g\|_\infty \leqslant 1$, the term (4.18) is bounded by

$$2u\,\mathrm{Lip}\, f\left(\max_{1\leqslant i\leqslant u}\mathbb{E}\left|X_{s_i}-X^{(r)}_{s_i}\right|+\max_{1\leqslant i\leqslant u}\mathbb{E}\left|X^{(r)}_{s_i}-\overline{X}^{(r)}_{s_i}\right|\right).$$

With the same arguments used in the proof of the existence of $H(Y^{(\infty)})$, the first term in the right-hand side of the inequality is bounded by

$$\sum_{i\geqslant s}2|i|b_i(\|Y_0\|_1+\|Y_0\|_{l+1}^{l+1}).$$

Notice now that if x,y are sequences with $x_i=y_i=0$ if $|i|\geqslant r$ then an infinitely repeated application of the previous inequality (2.4) yields

$$|H(x)-H(y)|\leqslant L(\|x\|_\infty^l\vee\|y\|_\infty^l\vee 1)\|x-y\| \tag{4.21}$$

where $L=\sum_{i\in\mathbb{Z}}b_i<\infty$ because $\sum_{i\in\mathbb{Z}}|i|b_i<\infty$. The second term is bounded by using eqn. (4.21)

$$\begin{aligned}\mathbb{E}\left|X^{(r)}_{s_i}-\overline{X}^{(r)}_{s_i}\right| &= \mathbb{E}\left|H\left(Y^{(r)}\right)-H\left(\overline{Y}^{(r)}\right)\right|\\ &\leqslant L\mathbb{E}\left(\left(\max_{-r\leqslant i\leqslant r}|Y_i|\right)^l\sum_{-r\leqslant j\leqslant r}|Y_j|\mathbb{1}_{Y_j\geqslant T}\right)\\ &\leqslant L(2r+1)^2\mathbb{E}\left(\max_{-r\leqslant i,j\leqslant r}|Y_i|^l|Y_j|\mathbb{1}_{|Y_j|\geqslant T}\right)\\ &\leqslant L(2r+1)^2\|Y_0\|_{m'}^{m'}T^{\ell+1-m'}\end{aligned}$$

The second term (4.19) of the sum is analogously bounded. The last term (4.20) can be written as

$$\left|\mathrm{Cov}\left(\overline{F}^{(r)}(Y_{s_i+j},1\leqslant i\leqslant u,|j|\leqslant r),\overline{G}^{(r)}(Y_{t_i+j},1\leqslant i\leqslant v,|j|\leqslant r)\right)\right|,$$

where $\overline{F}^{(r)}:\mathbb{R}^{u(2r+1)}\to\mathbb{R}$ and $\overline{G}^{(r)}:\mathbb{R}^{u(2r+1)}\to\mathbb{R}$. Under the assumption $r\leqslant[k/2]$, we use the $\varepsilon=\eta$ or λ-weak dependence of Y in order to bound this covariance term by $\psi\left(\mathrm{Lip}\overline{F}^{(r)},\mathrm{Lip}\overline{G}^{(r)},u(2r+1),v(2r+1)\right)\varepsilon_{k-2r}$, we set respectively $\psi(u,v,a,b)=uvab$ or $\psi(u,v,a,b)=uvab+ua+vb$.

$$\mathrm{Lip}\overline{F}^{(r)}=\sup\frac{|f(H(\overline{x}_{s_i+l},1\leqslant i\leqslant u,|l|\leqslant r)-f(H(\overline{y}_{s_i+l},1\leqslant i\leqslant u,|l|\leqslant r)|}{\sum_{j=1}^u\|x_j-y_j\|},$$

where the sup extends to $(x_1,\ldots,x_u)\neq(y_1,\ldots,y_u)$ where $x_i,y_i\in\mathbb{R}^{2r+1}$. Using eqn. (4.21)

$$\begin{aligned}|\overline{F}^{(r)}(x)-\overline{F}^{(r)}(y)| &\leqslant \mathrm{Lip}\, fL\sum_{i=1}^u\left(\|\overline{x}_{s_i}\|_\infty\vee\|\overline{y}_{s_i}\|_\infty\vee 1\right)^l\|\overline{x}_{s_i}-\overline{y}_{s_i}\|\\ &\leqslant \mathrm{Lip}\, fLT^l\sum_{i=1}^u\sum_{-r\leqslant l\leqslant r}|x_{s_i+l}-y_{s_i+l}|.\end{aligned}$$

We thus obtain $\mathrm{Lip}\, F^{(r)} \leqslant \mathrm{Lip}\, f \cdot L \cdot T^l$. Similarly $\mathrm{Lip}\, G^{(r)} \leqslant \mathrm{Lip}\, g \cdot L \cdot T^l$.
Under η-weak dependent inputs, we bound the covariance

$$|\operatorname{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leqslant (u\,\mathrm{Lip}\, f + v\,\mathrm{Lip}\, g)\Big\{4 \sum_{|i|\geqslant r} |i| b_i (\|Y_0\|_1 + \|Y_0\|_{l+1}^{l+1}) \\ + (2r+1)L\big((2r+1)2\|Y_0\|_{m'}^{m'} T^{l+1-m'} + T^l \eta_Y(k-2r)\big)\Big\}$$

We then fix the truncation $T^{m'-1} = \dfrac{2(2r+1)\|Y_0\|_{m'}^{m'}}{\eta_Y(k-2r)}$ to obtain the result of Lemma 3.2 in the η-weak dependent case.
Under λ-weak dependent inputs

$$|\operatorname{Cov}(f(X_{\mathbf{s}}), g(X_{\mathbf{t}}))| \leqslant (u\,\mathrm{Lip}\, f + v\,\mathrm{Lip}\, g + uv\,\mathrm{Lip}\, f\,\mathrm{Lip}\, g)\times \\ \times \Bigg(\Big\{4 \sum_{|i|\geqslant r} |i| b_i (\|Y_0\|_1 + \|Y_0\|_{l+1}^{l+1}) \\ + (2r+1)L\Big(2(2r+1)T^{l+1-m'}\|Y_0\|_{m'}^{m'} + T^l \lambda_Y(k-2r)\Big)\Big\} \\ \vee \Big\{(2r+1)^2 L^2 T^{2l} \lambda_Y(k-2r)\Big\}\Bigg)$$

We then set a truncation such that $T^{l+m'-1} = \dfrac{2\|Y_0\|_{m'}^{m'}}{L\lambda_Y(k-2r)}$ to obtain the result of Lemma 3.2 in the η-weak dependent case. □

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