Stochastic Simulation and Monte Carlo Methods: Exercise Class 2.

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Exercise 1. The truncated normal distribution is the one of $N \sim \mathcal{N}(0,1)$ given that $N > 0$.

- Determine the density of this distribution,
- Compute the efficiency M of the reject sampling from the proposal $Y \sim \mathcal{E}xp(1)$.

Exercise 2. The Beta distribution is given thanks to its density

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad x \in [0,1]\,.$$

Give a reject sampling of the distribution Beta(2,5) with proposal density $Y \sim \mathcal{U}(0,1)$.

Exercise 3. A random variable X has a $Gamma(\alpha,\beta)$ distribution, $\alpha > 0$ and $\beta > 0$, if it has the probability density function

$$f_{\alpha,\beta}(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)} \mathbb{1}_{x>0}.$$

Recall that $\forall \alpha > 0, \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

1. Show that $\forall \alpha > 0, \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. What is the value of $\Gamma(n)$, $n \in \mathbb{N}^*$?
2. What is the other name of a $Gamma(1,\beta)$ distribution?
3. Let X_1 and X_2 be independent random variables with distribution $Gamma(\alpha_1,\beta)$ and $Gamma(\alpha_2,\beta)$, and $c \in \mathbb{R}^{+*}$.
   (a) What is the distribution of $X_1 + X_2$?
   (b) What is the distribution of cX_1 ?
4. Let $U_1, \ldots, U_n$ be independent random variables with distribution $\mathcal{N}(0,1)$.
   (a) Show that U_1^2 has a $Gamma$ distribution and give its parameters. What is the other name of this distribution?
   (b) Same questions for $U_1^2 + \cdots + U_n^2$.
5. Show that if $E_1, \ldots, E_n, \ldots$ are iid $\mathcal{E}xp(\lambda)$ random variables, $\lambda > 0$, and $S_n = \sum_{i=1}^n E_i$ for any $n \in \mathbb{N}^*$, then $\inf\{n : S_{n+1} \geq 1\} \sim \mathcal{P}ois(\lambda)$. (This is a proposition of Chapter 3)

Exercise 4. Let (X, Y) be a standard gaussian vector in $\mathbb{R}^2$. Show that its radial part $R = X^2 + Y^2$ and its angular part $\Theta = \arctan(Y/X)$ are independent and distributed as $\mathcal{E}xp(1/2)$ and $\mathcal{U}nif(0, 2\pi)$.

Exercise 5. Show that the mixing distribution such that $X \mid Y \sim \mathcal{P}oisson(Y)$ with mixing variable $Y \sim \mathcal{G}amma(n, \lambda)$ with $n \geq 1$, $\lambda > 0$ is a negative binomial distribution of the form

$$\mathbb{P}(X = k) = \binom{k+n-1}{k} p^n (1-p)^k \qquad k \geq 0\,.$$

Determine p in function of λ and interpret X as the number of trials necessary for obtaining n successes in a specific experiment.

Exercise 6. Assume that $Z \sim \mathcal{B}(p)$, $p \in (0, 1)$, $X|Z = 0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$ and $X|Z = 1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ for μ_1, $\mu_2 \in \mathbb{R}$ and σ_1^2, $\sigma_2^2 > 0$.

1. What is the name of the distribution of X? Is it discrete or continuous?
2. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[X]$. Calculate the parameter $p \in (0, 1)$ given μ_1, μ_2 and $\mathbb{E}[X]$.
3. Compute $\mathrm{Var}\,(X)$ and comment.