Stochastic Modeling: Exercise Class 5.

Olivier Wintenberger: olivier.wintenberger@upmc.fr

Exercise 1. Assume we observe an iid sample $X_1, \dots, X_n \sim \mathcal{B}inomial(10, \theta)$ with $\theta \in [0, 1]$.

- Compute the log-likelihood $\log f(X_1, \dots, X_n \mid \theta)$.
- Calculate the MLE $\hat{\theta}_n$.
- Recognize the posterior distribution $f(\theta \mid X_1, \dots, X_n)$ from the a priori $\mathcal{B}eta(5, 3)$ distribution.
- Deduce the Bayes estimator $\hat{\theta}_n^B$.
- Express $\hat{\theta}_n^B$ as an aggregation of $\hat{\theta}_n$ and the mean of the prior distribution. Discuss the evolution of the weights with respect to n.

Exercise 2. Assume we observe an iid sample $X_1, \dots, X_n \sim \mathcal{N}(0, \theta)$ with $\theta > 0$.

- Calculate the MLE $\hat{\theta}_n$.
- Compute h, the posterior distribution $f(\theta \mid X_1, \dots, X_n)$ up to a multiplicative constant, from the a priori $\mathcal{G}amma(3, 5)$ distribution.
- Provide the Bayes estimator $\hat{\theta}_n^B$ as an integral.
- Describe an extended IS approximation of $\hat{\theta}_n^B$ with proposal $\mathcal{G}amma(c\hat{\theta}_n, c)$.
- What is the role of $c > 0$? How can we improve the choice of the proposal when n is small?

Exercise 3. Consider $I = \int_{-\infty}^{q} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx = 10^{-4}$ for q the quantile of order 10^{-4}

- Calculate the variance of the MC approximation with proposal $\mathcal{N}(0, 1)$.
- Provide a sufficiently large n so that the accuracy if the MC approximation is of the order of I.
- Calculate the variance of the IS approximation with proposal $\mathcal{N}(\mu, 1)$.
- Optimize its variance with respect to μ.
- Provide a sufficiently large n so that the accuracy if the IS approximation is of the order of I.