Stochastic Modeling: Exercise Class 7.

Olivier Wintenberger: olivier.wintenberger@upmc.fr

Exercise 1. Consider a game of coin tossing between two players A and B. If the toss results a head, then player B gives 1 euro to A, else player A gives 1 euro to B. Assume that the initial capital of A is a euros, B is b euros. We denote $p = \mathbb{P}(\text{"Head"})$.

- Model the gains of player A during the game as a discrete random walk (X_t) starting from $X_0 = 0$.
- Show that $\{0\}$ is an atom for (X_t) and compute $\mathbb{P}_0(X_k = 0)$ for $k \geq 1$.
- Show that $\eta_0(n)$ the number of passages at the atom $\sum_{t=1}^{n} \mathbb{1}_{\{X_t=0\}}$ is uniformly integrable if and only if $p \neq 1/2$.
- Consider the case $p > 1/2$. Show that $u_k = \mathbb{P}_k(\tau_0 = +\infty)$ satisfies the recursive equation $u_k = pu_{k+1} + (1-p)u_{k-1}$ for $k \geq 2$, $u_1 = pu_2$ and $\lim_{k\to\infty} u_k = 1$.
- Deduce that $\mathbb{P}_0(\tau_0 = \infty) = 2p - 1$ and discuss this unfair game.
- Consider the case $p = 1/2$. Show that

$$\mathbb{P}(X_{2t} = 0 | X_0 = 0) = \frac{(2t)!}{(t!)^2}\Big(\frac{1}{4}\Big)^t, \qquad t \geq 1\,.$$

- We recall Stirling's formula, valid for n large,

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n.$$

  Show that $\mathbb{E}[\eta_0(n)] \sim \sqrt{2n/\pi}$ and deduce that the Markov chain is null-recurrent.
- Show that ν the counting measure on $\mathbb{Z}$ is invariant. Explain why the distribution of (X_t) cannot converge to ν.