Time Series Analysis: TD2.

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Exercise 1. Let (Z_t) be a WN(1).

1. Show that if $|\phi| > 1$, the equation $X_t = \phi X_{t-1} + Z_t$ admits a second order stationary solution of the form $X_t = -\sum_{j=1}^{+\infty} \phi^{-j} Z_{t+j}$.
2. Show that it is a linear time series that is not causal.
3. Check that $\rho_X(h) = \phi^{-h}$, $h \geq 0$.
4. Show that the best linear prediction $\Pi_n(X_{n+1})$ coincides with $\phi^{-1}X_n$, $n \geq 1$.
5. Define Z'_t by the relation $Z'_t = X_t - \phi^{-1}X_{t-1}$. Express Z'_t in terms of Z_t.
6. Compute the variance of Z'_t.
7. Check that (Z'_t) is a WN.
8. Provide the causal linear representation (Wold) of (X_t).
9. **Application:** Let $X_t = 3X_{t-1} + Z_t$ with Z_t WN(σ^2) so that (X_t) is second order stationary with variance 1. We observe $(X_1, \ldots, X_5) = (0.6, 1.2, -0.6, 0.75, 0.27)$. What are the best linear predictions $\Pi_5(X_6)$ and $\Pi_5(X_7)$ of horizon 1 and 2 respectively?

Exercise 2. Let (Z_t) be a WN(σ^2) and the AR(2) model

$$X_t = \frac{8}{15}X_{t-1} - \frac{1}{15}X_{t-2} + Z_t \qquad t \in \mathbb{Z}\,.$$

1. We assume that X_t has variance 1. Find the Yule-Walker system relating $\gamma_X(1)$ and $\gamma_X(2)$.
2. Find the potential values of $\gamma_X(1)$ and $\gamma_X(2)$.
3. Find A and B so that

$$\frac{1}{(1-\frac{1}{3}z)(1-\frac{1}{5}z)} = \frac{A}{1-\frac{1}{3}z} + \frac{B}{1-\frac{1}{5}z}\,.$$

4. Deduce that the causal solution $X_t = \sum_{j\geq 0} \psi_j Z_{t-j}$ of this AR(2) model exists and compute the coefficients ψ_0, ψ_1 and ψ_2.
5. Compute $\Pi_n(X_{n+1})$ and $\Pi_n(X_{n+2})$ and their associated linear risks for $n \geq 2$ in function of σ^2.