Time Series Analysis: TD4.

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Exercise 1. Consider (Z_t) WN(σ^2) and the AR(p) model ($\phi_p \neq 0$)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t\,, \qquad t \in \mathbb{Z}.$$

1. Discuss the existence of a causal solution $X_t = \sum_{j\geq 0} \psi_j Z_{t-j}$ with $\sum_{j\geq 0} \psi_j^2 < \infty$.
2. Denote $\mathbb{X}_t = (X_t, \ldots, X_{t-p+1})^\top \in \mathbb{R}^p$ and show that $(\mathbb{X}_t)$ is an homogeneous Markov chain of the form $\mathbb{X}_t = \mathbf{A}\mathbb{X}_{t-1} + \mathbb{Z}_t$, $t \geq 0$. Identify $\mathbf{A}$ and $\mathbb{Z}_t$.
3. Show that $z^p\phi(z^{-1})$ with $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ is the characteristic polynomial of $\mathbf{A}$.
4. Deduce that the largest eigenvalue $\rho(\mathbf{A})$ of $\mathbf{A}$ has modulus smaller than 1 if and only if the roots of ϕ are outside the unit disc.
5. Show that
$$\mathbb{X}_t = \sum_{j\geq 0} \mathbf{A}^j \mathbb{Z}_{t-j}\,, \qquad t \in \mathbb{Z}\,,$$
and that the series is normally convergent in $\mathbb{L}^2$.
6. Deduce an algorithm to compute the coefficients ψ_j using the $\mathbf{A}$ and check that they decrease exponentially fast.

Exercise 2. We have the following theorem

Theorem 1 (Hannan)**.** *If (X_t) satisfies an ARMA(p,q) model $\phi(L)X_t = \gamma(L)Z_t$ with $\theta = (\phi_1, \ldots, \phi_p, \gamma_1, \ldots, \gamma_q)^\top \in \mathcal{C}$ and (Z_t) SWN(σ^2), $\sigma^2 > 0$, then the QMLE is asymptotically normal*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d.} \mathcal{N}_{p+q}\left(0, Var\,(AR_p, \ldots, AR_1, MA_q, \ldots, MA_1)^{-1}\right)$$

where (AR_t) and (MA_t) are the stationary AR(p) and AR(q) satisfying

$$\phi(L)AR_t = \eta_t, \qquad \gamma(L)MA_t = \eta_t, \qquad t \in \mathbb{Z},$$

with the same SWN(1) (η_t).

Let us consider the ARMA(1,1) model

$$X_t = \phi X_{t-1} + Z_t + \gamma Z_{t-1},$$

with $\theta = (\phi, \gamma)^\top$.

1. Determine the causal representation of the time series (AR_t) and (MA_t) under the appropriate condition on θ.
2. Compute the matrix of variance-covariance $\mathrm{Var}(AR_1, MA_1)$.
3. Determine the assumption for inverting the matrix and interpret it.
4. Invert the matrix.
5. Deduce a consistent estimator of the asymptotic variance by plugging in the QMLE $\hat{\theta}_n$.
6. Deduce a confidence region of asymptotic level $1 - \alpha$ for the coefficient θ.